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Closed loop transfer functions and internal stability

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By a polynomial formulation, we explicitely and exhaustively define the set of tracking and regulation transfer functions a process can assume by linear closed loop control under constraint of internal stability. This result exhibits the freedom let to control law designers and allows us to study new results on adaptive control in a unified framework.

1 - Introduction

An important question of process controls is to define the set of closed loop transfer functions a given process can assume while meeting some constraints such as internal stability. This question has been considered in numerous studies but more explicitely in the model following problem (MFP). Among the works we were aware of, let us mention that of Morse who gives an implicit condition for the MFP solution through a geometric approach [1]. When this condition is met, the model is said to be admissible and if more over internal stability (IS) is satisfied then the model is said to be acceptable. In [2, Th 8.5.2], using a polynomial approach, Wolovich shows that the MFP is equivalent to that of finding a causal transfer function, solution of a rational matrix equation. Forney in [3, Th6] gives then a necessary and sufficient condition for admissibility, equivalent to that of Morse. A more explicit result exists in [4,Th.4.5] for a right invertible process and is completed in [5] as for acceptability. More complete results can be found in [6] where Youla et al. give a necessary and sufficient polynomial condition for the MFPRIS (model following problem with robust internal stability) without input dynamics. And finally, after the first writting of this paper, Pernebo gave in [7],[8] complete and explicit results using Λ-generalized polynomials. Our approach here is more restrictive but it better exhibits the key role played by unstable open loop poles and zeros.

Using a polynomial formulation, we here give an explicit and complete description of admissible and acceptable tracking and regulation models for both MFPIS and MFPRIS. In chapter II, we set our problem, chapter III is devoted to some technical lemmas, the main results and comments can be found in chapter IV and chapter V is that of conclusions. All proofs are omitted (see [9]).

2 - Problem statement

Process Let us consider a $(l\times m)$ linear, rational, time invariant plant we assume to be completely controllable and to be described by

$$A_1(b)z_k = u_k \quad , \quad y_k = B_1(b)z_k + w_k \tag{1}$$

where u_k, z_k in R^m are input and partial state vectors, y_k, w_k in R^l are output and disturbance vectors, and $A_1(b)$, $B_1(b)$ are $(m\times m)$ and $(l\times m)$ polynomial matrices in b such that b is the backward shift operator $(bu_k = u_{k-1})$ and :

$A_1(0)$ is invertible (causality), Rank $B_1(b)=n$ (HP)

Control law : we consider here the most general linear rational time invariant control law given by :

$$C(b)u_k = R(b)v_k - D(b)y_k \tag{2}$$

where v_k in R^l is set point vector and $C(b)$, $D(b)$, $R(b)$ are $m\times m$, $m\times l$ and $m\times l$ polynomial matrices such that :

$C(0)$ invertible, $\lim_{b\to 0} \frac{C^{-1}(b)R(b)}{b}$ and $\lim_{b\to 0} \frac{C^{-1}(b)D(b)}{b}$ finite (HC)

closed loop transfer functions : we have six closed loop transfer functions

$$\begin{aligned}
U_t(b) &= A_1(b)E^{-1}(b)R(b) & U_r(b) &= -A_1(b)E^{-1}(b)D(b)\\
Z_t(b) &= E^{-1}(b)R(b) & Z_r(b) &= -E^{-1}(b)D(b) \quad (3)\\
Y_t(b) &= B_1(b)E^{-1}(b)R(b) & Y_r(b) &= I-B_1(b)E^{-1}(b)D(b)
\end{aligned}$$

$$E(b)=C(b)A_1(b) + D(b)B_1(b) \tag{4}$$

where U,Z,Y specify input, partial state and output respectively while indices t and r specify set point and disturbance.

problem statement: Let $\mathcal{L}$ be a subset of C containing the origin (usually $\mathcal{L}$ is the unit disc), let $\mathcal{M}_{p\times q}$ be the set of p×q rational matrices, $T(b)$ in $\mathcal{M}_{p\times q}$ is said to be $\mathcal{L}$-stable iff the determinant of any minimal representation (see[10]) has all its roots strictly out of $\mathcal{L}$, $T(b)$ is causal iff $\lim_{b\to\infty} T(b)$ finite and strictly causal iff $\lim_{b\to\infty} \frac{T(b)}{b}$ finite.

We want to completely characterize the following three sets :

(i) $\mathcal{M}_1$ = set of $M_t(b)$, $M_r(b)$ in $\mathcal{M}_{l\times l}$ such that a control law (C,D,R) exists meeting conditions (HC) and with (3) :

$$M_t(b) = Y_t(b), \; M_r(b) = Y_r(b)$$

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$\mathcal{M}_1$ is the set of admissible tracking and regulation models for the MFP.
(ii) $\mathcal{M}_2$ = subset of $\mathcal{M}_1$ such that all the transfer functions of (3) are stable. $\mathcal{M}_2$ is the set of acceptable tracking and regulation models for the MFPIS.
(iii) $\mathcal{M}_3$ = subset of $\mathcal{M}_2$ such that it exists a neighbourhood of A_1, B_1, such that any pair $(\bar A_1, \bar B_1)$ belonging to this neighbourhood and meeting conditions (HP) gives stable transfer functions in (3). $\mathcal{M}_3$ is the set of acceptable tracking and regulation models for the MFPRIS and is the only set making sense when initial conditions of partial state are unknown.

3 - Technical lemmas

Lemma 1: (Key lemma) Any (l×m) polynomial matrix $B(b)$ with rank n may be split into :

$$B(b) = B_u(b)\, B_s(b) \qquad (5)$$

where $B_u(b)$ is an (l×n) left invertible polynomial matrix whose zeros (complex value for which rank is decreased) lie inside $\mathcal{L}$, and $B_s(b)$ is an (n×m) right invertible polynomial matrix whose zeros lie strictly outside $\mathcal{L}$. B_u and B_s are defined up to respectively post and pre multiplication by a unimodular matrix.

This lemma exhibits two properties of polynomial matrices as linear operator they admit a factorization into epic and monic parts, as polynomials they admit a factorization into their divisors.

Notations : For given polynomial matrices $A_1(b)$ and $B_1(b)$, omitting the argument b and making respective use of lemma 1, left minimal representation, lemma 1 (dual), left minimal representation and left minimal representation, we note :

$$B_1 A_1^{-1} = (B_{1u} B_{1s}) A_1^{-1} = B_{1u}(\bar A^{-1} B_s) = B_{1u}(\bar A_s \bar A_u)^{-1} B_s$$
$$= (A_u^{-1} \bar B_u) \bar A_s^{-1} B_s = A_u^{-1}(A_s^{-1} B_u) B_s = A^{-1} B \qquad (6)$$

Lemma 2: A necessary and sufficient condition for $A_1(b)$, $B_1(b)$ to be detectable (stable greatest common right divisor) is that $B_{1u}(b)$, $\tilde A_u(b)$ be relatively right prime. Then, after [10], it exits polynomial matrices X, Y, X_1, Y_1 such that the two following block matrices are unimodular and inverse of each other :

$$\begin{pmatrix} X_1 & Y_1 \\ -\tilde B_u & A_u \end{pmatrix} \begin{pmatrix} \tilde A_u & -Y \\ B_{1u} & X \end{pmatrix} = I \qquad (7)$$

4 - Main Results and Comments

Let $\bar{\mathcal{M}}_{p\times q}$ be the subset of $\mathcal{M}_{p\times q}$ of strictly causal matrices and $\tilde{\mathcal{M}}_{p\times q}$ the subset of $\bar{\mathcal{M}}_{p\times q}$ of $\mathcal{L}$-stable matrices, we have :

Theorem 1 : If $B^{o}_{1u}(b)$ is obtained from $B_{1u}(b)$, with $\mathcal{L}$ being reduced to the origin, then

$$\mathcal{M}_1 = \left\{ B^{o}_{1u}(b)\, \bar M_t(b),\ I - B^{o}_{1u}(b) \bar M_r(b) / \bar M_t(b),\ \bar M_r(b) \in \bar{\mathcal{M}}_{n\times l} \right\} \quad (8)$$

Theorem 2 :

$$\mathcal{M}_2 = \left\{ B_{1u}(b) \bar M_t(b),\ I - B_{1u}(b) \bar M_r(b) / \bar M_t(b),\ \bar M_r(b) \in \bar{\mathcal{M}}_{n\times l} \right\} \quad (9)$$

Theorem 3 : $\mathcal{M}_3$ is nonempty iff $A_1(b)$, $B_1(b)$ is detectable and :

$$\mathcal{M}_3 = \Big\{ B_{1u}(b) \tilde M_t(b),\ [X(b) - B_{1u}(b) \tilde M_r(b)] A_u(b),$$
$$/ \tilde M_t(b),\ \tilde M_r(b) \in \tilde{\mathcal{M}}_{n\times l} \Big\} \qquad (10)$$

Comments : 1-Theorem 1 shows that the output has to be in Span$[B^{o}_{1u}(b)]$ even if stability is not required. Thus not only process delays but also geometric aspect of process right non invertibility make problems. The left invertible operator $B^{o}_{1u}(b)$ extends the interactor notion (see[4]).

2-Theorem 2 completes theorem 3.1. of [1] : the closed loop poles are : tracking and regulation reference model poles, a part of open loop zeros and a free set of zeros (in Ker$[B_{1s}(b)]$). Moreover open loop zeros are distributed between closed loop zeros and closed loop poles. Thus a necessary condition for stability is to keep unstable open loop zeros as closed loop zeros. Furthermore, we have the following property : If the output is in Span$[B_{1u}(b)]$ with a bounded antecedent, if the partial state projection on Ker$[B_{1s}(b)]$ is bounded, then input, partial state and output are all bounded. This result allows for an extension of the results in [11] to direct adaptive control of left invertible process with only delays as unstable zeros and known operator $B^{o}_{1u}(b)$.

3-According to theorem 3 open loop poles are distributed between zeros of matrix E(b) in (4) and regulation zeros. Thus a necessary condition for robust stability is to keep unstable open loop poles as regulation zeros. Note that robustness affects only the regulation transfer function and that there is no robustness problem for stable processes (see[12]).

4-Theorem 3 is a restriction of Th.4.7 of [7] and Th.5.1 of [8] in the case when controlled outputs are measured and disturbances are directly additive on outputs. Moreover when $A_1(b)$ and $B_1(b)$ are relatively right prime, using (7) one can show the equivalence of the part of this theorem dealing with regulation, with lemma 3 of [6].

control law design : Omitting argument b, let $F^{-1}G$, $H^{-1}J$ be minimal representations of $\tilde M_t$, $\tilde M_r$ in $\tilde{\mathcal{M}}_{n\times l}$, let $K^{-1}L$ be a minimal representation of FH^{-1}. With (7) we define P, Q as follows :

$$(P \quad Q) = (H \quad J) \begin{pmatrix} X_1 & Y_1 \\ -\tilde B_u & A_u \end{pmatrix} \qquad (11)$$

Then let N be a greatest common left divisor of LQ, K G with respectively $\bar D$, $\bar R$ as quotients and with (6), if $C^{-1}\tilde D$ is a minimal representation of $B_s^{+} \tilde A_s P^{-1} L^{-1} N$, where B_s^{+} is a stable right inverse of B_s, we meet equation (11) by taking

$$R = \tilde D \bar R, \qquad D = \tilde D \bar D \qquad (12)$$

A FORTRAN programm has been written to realize this design [13], it uses exact polynomial computation via Hensel codes.

Thus control law design may be reduced to finding stable and strictly causal rational transfer functions such that the closed loop transfer functions are the "best" possible. For this "best" to make sense we may use Wiener-Hopf approach to minimize a quadratic distance between desired and acceptable behaviours which with (10) are completely defined in terms of unstable open loop poles and zeros and the monic factor $B^o_{1u}(b)$. Then compared with [6] we may expect simplification of this minimization due to order reduction.
Obviously this task is too much computation demanding to be achieved on line for adaptive control. In this case the simplest technique is to use directly equation (4) for pole placement but without well defined zero placement [14], [15].

5 - Conclusion

We have answered the question asked by Morse in [1]: The set of regulation and tracking closed loop transfer functions a plant can assume while preserving internal stability is explicitely and exhaustively characterized in terms of unstable open loop poles and zeros and a monic factor of a controllable polynomial right representation. This naturally exhibits the freedom one has in designing control laws and shows that one can work with these irreducible components only, whereas any linear time invariant control law can be reached in this unified framework.

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