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Prediction-Based Control for Linear Systems with Input- and State- Delay – Robustness to Delay Mismatch

D. Bresch-Pietri * **and N. Petit** **

* *Massachusetts Institute of Technology, Dpt. of Mechanical Engineering, 77 Massachusetts Avenue, Cambridge MA, 02139, USA (e-mail : dbp@mit.edu)*
** *MINES ParisTech, Centre Automatique et Systèmes, Unité Mathématiques et Systèmes, 60 Bd St-Michel, 75272 Paris, Cedex 06, France*

Abstract: This paper addresses the design of a robust prediction-based controller for linear systems with both input and state delays. We extend the usual prediction-based scheme to state delay and prove its robustness to sufficiently small delay mismatches. Our approach is grounded on the linking of two recently proposed infinite-dimensional techniques: a Complete-Type Lyapunov functional, which enables state delay systems stability analysis, and tools from the field of Partial Differential Equations, reformulating the delays as transport equations and introducing a tailored backstepping transformation. We illustrate the merits of the proposed technique with simulations on a process dryer system.

1. INTRODUCTION

Predictor-based control strategies, more commonly known as Smith Predictor (see Smith [1959], Artstein [1982], Manitius and Olbrot [1979]) are state-of-the-art for systems with constant input time-delays (see for instance Gu and Niculescu [2003], Jankovic [2008], Michiels and Niculescu [2007], Bresch-Pietri et al. [2012], Bekiaris-Liberis and Krstic [2013] or Richard [2003] and the reference therein). This technique is grounded on the use of a prediction of the system state on a time horizon equal to the input delay and aims at compensating this delay, which notably improves the transient performances.

Another class of delay systems which are commonly employed, e.g., to model population dynamics (see Erneux [2009], Ruan [2006]) or process systems with recycle loops (see Meyer et al. [1978], Chauvin et al. [2007], Bresch-Pietri et al. [2013]), are dynamics involving an additional state delay. However, while the literature on control of either input delay or state delay systems is extremely wealth, systems with both input and state delays have seldom been studied.

In this paper, we focus on the design of prediction-based controller for linear systems with both input and state delay. Unlike the predictor feedback for a problem with input delay only, in this problem, the prediction employs an infinite dimensional semi-group in the distributed part of the feedback. This semi-group cannot anymore be written explicitly. To counteract this difficulty, we extend recent result for stability analysis of state delay systems (see Kharitonov [2013b]) to propose a corresponding Lyapunov analysis and study the robustness of the controller to delay mismatch.

Our approach is grounded on new tools that were proposed lately to address input delay uncertainties (see Krstic [2008a], Krstic [2008b]). This methodology is based on a modeling of the actuator delay as a transport Partial Differential Equation (PDE) coupled with the original Ordinary Differential Equation (ODE) and on a corresponding suitable backstepping transformation. We extend this framework for state delays, by introducing an additional transport PDE accounting for the state delay and proposing a new backstepping transformation. This is the main contribution of the paper.

For the sake of clarity, we consider constant delay estimates and only investigate the robustness of a prediction-based controller to delays mismatches. Further, we address the two robustness problems separately, by considering first state delay uncertainties and then input delay uncertainties respectively. However, from the presented elements, a careful reader can deduce that existing delay-adaptive techniques involving time-varying estimates (see Bresch-Pietri and Krstic [2010], Bresch-Pietri et al. [2012], Bresch-Pietri [2012]) for both delays in the same time can be straightforwardly applied within this new framework. The objective of this paper is to present these new tools.

The paper is organized as follows. In Section 2, we introduce the problem under consideration and some properties of interest in the sequel. We present the design methodology that we propose to employ in Section 3 and apply it to address robustness to state delay in Section 4 and to input delay in Section 5. Finally, we illustrate the merits of our results in Section 6 with simulations of a dryer process involving a recycle loop.

Notations In the following, n and p are strictly positive integers, $|.|$ refers to the usual Euclidean norm whereas the norm $\|.\|$ is the spatial $\mathscr{L}_2$-norm defined as

$$\|f(t)\| = \sqrt{\int_0^1 |f(x,t)|^2 dx}\,,\; f : [0;1] \times \mathbb{R}_+ \to \mathbb{R}^p$$

For $D_1 \geq 0$, we write $X_t : s \in [-D_1, 0] \mapsto X(t+s) \in \mathbb{R}^n$ the restriction of the state X on the interval $[t-D_1, t]$ and, for $D \geq 0$, we write $U_t : s \in [-D, 0] \mapsto U(t+s) \in \mathbb{R}$ the input restriction on the interval $[t-D, t]$. We write $\partial_x f$ the partial derivative of a function f with respect to a variable x. We write $\lambda_m(M)$ the minimal eigenvalue of a given matrix $M \in \mathscr{M}_n$. Finally, we denote $\mathscr{C}_{pw}(S_1, S_2)$ the set of piecewise continuous functions defined on the set S_1 and taking values in S_2.

2. PROBLEM STATEMENT AND PRELIMINARIES

2.1 Control objective

Consider the potentially unstable system

$$\dot{X}(t) = A_0X(t) + A_1X(t-D_1) + BU(t-D) \tag{1}$$

in which $X \in \mathbb{R}^n$, U is scalar and $D \geq D_1 > 0$. We consider that the state delay $D_1 \in [\underline{D}_1, \overline{D}_1]$ ($\underline{D}_1 > 0$) and the input delay $D \in [\underline{D}, \overline{D}]$ ($\underline{D} > 0$) are uncertain. The control objective is to stabilize the plant through a prediction-based methodology, despite delay uncertainties. With this aim in view, we first formulate the following assumption.

Assumption 1. There exists $(K_0, K_1) \in (\mathbb{R}_{1,n})^2$ such that the system

$$\dot{X}(t) = (A_0 + BK_0)X(t) + (A_1 + BK_1)X(t-D_1) \tag{2}$$

is exponentially stable.

This assumption states that, for a given state delay D_1, there exist gains such that a delayed plant in which the input delay is compensated but not the state delay is exponentially stable. Determination of such feedback gains is beyond the scope of this paper, but may reveal to be a challenging task involving Linear Matrix Inequalities (LMIs) resolution (see Gu et al. [2003]). Depending on the algebraic structure of the plant, it may even be possible to compensate the state delay, i.e. to choose K_1 such that $A_1 + BK_1 = 0$ (see the strict-feedback system studied in Bekiaris-Liberis and Krstic [2010] and the scalar example provided in Section 6).

Intuitively, extending straightforwardly the usual prediction-based methodology (see Artstein [1982], Manitius and Olbrot [1979]) to the case of the dynamics (1) yields the control choice $U(t) = K_0X(t+D) + K_1X(t+D-D_1)$. However, determination of expressions of these predictions involves to integrate the dynamics (1), which cannot be done explicitly. This is the point that we now address.

2.2 Preliminaries: properties and nominal design

In this section, we present some elements of interest for the sequel of the paper.

Proposition 1. (*Bellman and Cooke [1963]*). Consider $\Phi(\cdot, D_1)$ the transition matrix of system (1), which satisfies the differential equation

$$\begin{cases} \partial_t\Phi(t,D_1) = A_0\Phi(t,D_1) + A_1\Phi(t-D_1,D_1),\ t \geq 0 \\ \Phi(0,D_1) = I,\ \Phi(s,D_1) = 0 \text{ for } s \in [-D_1, 0[\end{cases} \tag{3}$$

and, besides, the following one

$$\begin{cases} \partial_t\Phi(t,D_1) = \Phi(t,D_1)A_0 + \Phi(t-D_1,D_1)A_1,\ t \geq 0 \\ \Phi(0,D_1) = I,\ \Phi(s,D_1) = 0 \text{ for } s \in [-D_1, 0[\end{cases} \tag{4}$$

Then, given an initial time t_0 and initial conditions $X_{t_0} \in \mathscr{C}_{pw}([-D_1,0],\mathbb{R}^n)$ and $U_{t_0} \in \mathscr{C}_{pw}([-D,0],\mathbb{R})$, the corresponding solution of system (1) writes, for $t_1 \geq t_0$,

$$X(t_1) = \Phi(t_1-t_0,D_1)X(t_0) + \int_{t_0-D}^{t_1-D} \Phi(t_1-D-s,D_1)BU(s)ds + \int_{t_0-D_1}^{t_0} \Phi(t_1-s-D_1,D_1)A_1X(s)ds \tag{5}$$

In particular, this result enables one to compute the system state predictions $X(t+D)$ and $X(t+D-D_1)$ in terms of past values of the system state X_t and of the input U_t. This gives rise to the following result.

Proposition 2. (*Kharitonov [2013a]*). Given initial conditions $X_0 \in \mathscr{C}_{pw}([-D_1,0],\mathbb{R}^n)$ and $U_0 \in \mathscr{C}_{pw}([-D,0],\mathbb{R})$, then the closed-loop system consisting of the plant (1) with $D_1 \leq D$ and the control law

$$U(t) = K_0X(t+D) + K_1X(t+D-D_1) \tag{6}$$

$$\begin{aligned} &= \left[K_0\Phi(D,D_1) + K_1\Phi(D-D_1,D_1)\right]X(t) \\ &\quad + \int_{t-D_1}^{t} \left[K_0\Phi(t+D-D_1-s,D_1) + K_1\Phi(t+D-2D_1-s,D_1)\right]A_1X(s)ds \\ &\quad + \int_{t-D}^{t} K_0\Phi(t-s,D_1)BU(s)ds \\ &\quad + \int_{t-D}^{t-D_1} K_1\Phi(t-D_1-s,D_1)BU(s)ds \end{aligned} \tag{7}$$

is exponentially stable, in the sense that there exists $\gamma, \mu > 0$ such that, for $t \geq 0$,

$$|X(t)| + |U(t)| \leq \mu \left(\max_{s\in[-D_1,0]} |X_0(s)| + \max_{s\in[-D,0]} |U(s)| \right) e^{-\gamma t}$$

Intuitively, one can understand this result observing that, plugging the control law (6), for $t \geq D$ into (1), the resulting closed loop system should be (2) which is exponentially stable according to Assumption 1. Determination of the explicit control law (7) follows directly from (5) with $(t_1,t_0) = (t+D,t)$ and $(t_1,t_0) = (t+D-D_1,t)$ respectively [1] [2]. However, proving stability without exponential estimates for the entire system state (meaning in terms of X_t and U_t and for all $t \geq 0$) is not an easy task and was only provided recently in Kharitonov [2013a].

As a final ingredient to be used in the sequel, we reformulate (1) with transport PDEs as

$$\begin{cases} \dot{X}(t) = A_0X(t) + A_1\zeta(0,t) + Bu(0,t) \\ \begin{cases} D_1\partial_t\zeta(x,t) = \partial_x\zeta(x,t) \\ \zeta(1,t) = X(t) \end{cases} \quad \begin{cases} D\partial_t u(x,t) = \partial_x u(x,t) \\ u(1,t) = U(t) \end{cases} \end{cases} \tag{8}$$

in which we have introduced the distributed system state $\zeta(x,t) = X(t+D_1(x-1))$ and the distributed input $u(x,t) = U(t+D(x-1))$, $x \in [0,1]$ and $t \geq 0$. In details, with this new representation, the system consists of an ODE driven by the outputs of two transports PDE, one fed by the current system state $X(t)$ propagated with speed $1/D_1$ and the other fed by the current control $U(t)$ propagated with speed $1/D$. These two transport equations account respectively for the state and input delays.

We are now ready to focus on the control design in case of delays uncertainties. For the sake of clarity of exposition, we deal with these difficulties separately in the sequel.

[1] Note that the control law (6) would also hold in the case $D \leq D_1$, but that in this case $X(t+D-D_1)$ is a past value of the system state and not a prediction, as $t+D-D_1 \leq t$. Consequently, the controller equation (7) does not hold as $\Phi(s,D_1) = 0$ for $s \in [-D_1, 0)$ and neither does the analysis provided in the sequel. However, this case directly falls into the context of input-delay compensation and can be addressed with the tools previously proposed in Krstic [2008a].

[2] Note also that this control law, defined through an integral equation, is well-defined. Indeed, it can be re-written as $U(t) = f(t) + \int_{-D}^{t} K(t,s)U(s)ds$ which is a second-order Volterra equation, with $K(t,s) = 0$ for $s < t-D$ and $K(t,s) = [K_0\Phi(t-s,D_1) + K_1\Phi(t-D_1-s,D_1)]B$ otherwise. Following Polyanin and Manzhirov [2007], the solution of this equation exists and is unique.

3. PROPOSED METHODOLOGY

To account for delay uncertainties, we propose to introduce distributed estimates corresponding to the transport variables introduced in (8), with estimated speed of propagation fed by the delay estimates. These distributed estimates are suitably scaled to enable comparison between their propagation speed.

To analyze the effect of these errors on the closed-loop stability, we transform these variables via the backstepping technique (see Krstic [2008a], Krstic and Smyshlyaev [2008]). This enables to reformulate the plant in the form of a stable ODE, fed by transport equations with source terms with vanishing boundary conditions. This equivalent representation is then used to provide sufficient conditions for stabilization via a Lyapunov analysis.

Finally, we take advantage of the complete-type Lyapunov-Krasovskii functional introduced in Kharitonov and Zhabko [2003] to build Lyapunov functional candidates and develop a stability analysis. This is the methodology applied in the sequel.

4. ROBUSTNESS TO STATE DELAY MISMATCH

In this section, we consider that the input delay D is perfectly known, while the state delay $D_1 \in [\underline{D}_1, \bar{D}_1]$ is uncertain. Correspondingly, we consider a constant delay estimate $\hat{D}_1 \in [\underline{D}_1, \bar{D}_1]$.

4.1 Control design

Following the certainty equivalence principle applied to the nominal controller (7), we choose the control law

$$\begin{aligned} U(t) &= [K_0\Phi(D,\hat{D}_1) + K_1\Phi(D-\hat{D}_1,\hat{D}_1)]X(t) \\ &+ \int_{t-\hat{D}_1}^{t} [K_0\Phi(t+D-\hat{D}_1-s,\hat{D}_1) \\ &\qquad + K_1\Phi(t+D-2\hat{D}_1-s,\hat{D}_1)]A_1X(s)ds \\ &+ \int_{t-D}^{t} K_0\Phi(t-s,\hat{D}_1)BU(s)ds \\ &+ \int_{t-D}^{t-\hat{D}_1} K_1\Phi(t-\hat{D}_1-s,\hat{D}_1)BU(s)ds \end{aligned} \quad (9)$$

Theorem 1. Consider the closed-loop system consisting of the plant (1) and the control law (9). Define the functional

$$\Gamma(t) = |X(t)|^2 + \int_{t-\max\{D_1,\hat{D}_1\}-\hat{D}_1}^{t} |X(s)|^2 ds + \int_{t-D-\hat{D}_1}^{t} U(s)^2 ds \quad (10)$$

There exists $\delta^* > 0$ such that, if $|D_1 - \hat{D}_1| < \delta^*$, there exists $R > 0$ and $\rho > 0$ such that

$$\Gamma(t) \leq R\Gamma(0)e^{-\rho t}, \; t \geq 0 \quad (11)$$

This result states that exponential stabilization is preserved, provided that the state delay estimation error is sufficiently small. Even if an expression of δ^* is proposed in the section below, we do not aim here at providing a quantitative bound for this critical error but only at presenting a proof for a seemingly intuitive result.

4.2 Proof of Theorem 1

Before detailing the Lyapunov stability analysis for the closed-loop system, we introduce several intermediate variables which are used to reformulate the plant in a more suitable form.

Distributed estimates To take into account the effects of the state delay estimation, we introduce the following distributed state estimate with the corresponding distributed state error

$$\hat{\zeta}(x,t) = X(t + \hat{D}_1(x-1)) \quad (12)$$

$$\tilde{\zeta}(x,t) = \zeta(x,t) - \hat{\zeta}(x,t) \quad (13)$$

Lemma 1. Define $\tilde{D}_1 = D_1 - \hat{D}_1$. The distributed states defined in (12)–(13) satisfy the dynamics

$$\begin{cases} \hat{D}_1\partial_t\hat{\zeta} = \partial_x\hat{\zeta} \\ \hat{\zeta}(1,t) = X(t) \end{cases} \qquad \begin{cases} D_1\partial_t\tilde{\zeta} = \partial_x\tilde{\zeta} - \dfrac{\tilde{D}_1}{\hat{D}_1}\partial_x\hat{\zeta} \\ \tilde{\zeta}(1,t) = 0 \end{cases} \quad (14)$$

Proof: Taking a time- and spatial-derivative of the distributed state estimate (12), one obtains that it satisfies $\hat{D}_1\partial_t\hat{\zeta} = \partial_x\hat{\zeta}$. Consequently, using (8), it follows that

$$D_1\partial_t\tilde{\zeta} = \partial_x\zeta - \frac{D_1}{\hat{D}_1}\partial_x\hat{\zeta} = \partial_x\tilde{\zeta} - \frac{\tilde{D}_1}{\hat{D}_1}\partial_x\hat{\zeta}$$

and the boundary condition can be obtained noticing that $\zeta(1,t) = \hat{\zeta}(1,t) = X(t)$. ∎

Lemma 2. Define the distributed variables

$$\begin{aligned} \hat{z}(x,t) &= \Phi((D-\hat{D}_1)x,\hat{D}_1)X(t) \\ &+ \hat{D}_1\int_0^1 \Phi((D-\hat{D}_1)x - \hat{D}_1y,\hat{D}_1)A_1\hat{\zeta}(y,t)dy \\ &+ D\int_0^{\frac{D-\hat{D}_1}{D}x} \Phi((D-\hat{D}_1)x - Dy,\hat{D}_1)Bu(y,t)dy \end{aligned} \quad (15)$$

$$\hat{v}(x,t) = \begin{cases} \hat{\zeta}\left(\frac{D}{\hat{D}_1}x,t\right) & \text{for } x \in [0,\hat{D}_1/D) \\ \hat{z}\left(\frac{Dx-\hat{D}_1}{D-\hat{D}_1},t\right) & \text{for } x \in [\hat{D}_1/D,1] \end{cases} \quad (16)$$

The following dynamics is satisfied, for $x \in [0,1]$ and $t \geq 0$,

$$\begin{cases} D\partial_t\hat{v} = \partial_x\hat{v} + D\Phi(Dx-\hat{D}_1,\hat{D}_1)A_1\tilde{\zeta}(0,t) \\ \hat{v}(0,t) = \hat{\zeta}(0,t) \end{cases} \quad (17)$$

In this lemma, we introduce a prediction $z(x,t)$ of the system state at time $t + x(D-\hat{D}_1)$, starting from the current system state $X(t)$. Concatenating z with the previous distributed state ζ, we aim at estimating the system state over the whole time interval $[t-\hat{D}_1, t+D-\hat{D}_1]$.

Proof: We first highlight the fact that the transport equation is indeed a transport equation in the case of perfect delay knowledge as $\hat{v}(\hat{D}_1/D) = \hat{\zeta}(1,t) = \hat{z}(0,t) = X(t)$. Now, taking spatial- and time-derivative of (15) and using the dynamics and initial condition of the transition matrix Φ, one gets

$$\begin{aligned} \partial_t\hat{z} &= \Phi((D-\hat{D}_1)x,\hat{D}_1)\left[A_0X(t) + A_1X(t-D_1) + BU(t-D)\right] \\ &+ \hat{D}_1\int_0^1 \Phi((D-\hat{D}_1)x - \hat{D}_1y,\hat{D}_1)A_1\partial_t\hat{\zeta}(y,t)dy \\ &+ D\int_0^{\frac{D-\hat{D}_1}{D}x} \Phi((D-\hat{D}_1)x - Dy,\hat{D}_1)B\partial_t u(y,t)dy \end{aligned}$$

$$\begin{aligned} \partial_x\hat{z} &= (D-\hat{D}_1)\Bigg[\Phi((D-\hat{D}_1)x,\hat{D}_1)A_0X(t) \\ &+ \int_0^1 \Phi((D-\hat{D}_1)x - \hat{D}_1y,\hat{D}_1)A_1\partial_x\hat{\zeta}(y,t)dy \\ &+ \Phi((D-\hat{D}_1)x,\hat{D}_1)A_1\hat{\zeta}(0,t) + \Phi((D-\hat{D}_1)x,\hat{D}_1)Bu(0,t) \\ &+ \int_0^{\frac{D-\hat{D}_1}{D}x} \Phi((D-\hat{D}_1)x - Dy,\hat{D}_1)B\partial_x u(y,t)dy\Bigg] \end{aligned}$$

in which we have used integrations by parts. Using the transport PDEs in (8)–(14) and the facts that $\hat{\zeta}(0,t) = X(t-D_1) - \tilde{\zeta}(0,t)$ and that $u(0,t) = U(t-D)$, one obtains that, for $x \in [0,1]$,

$$(D-\hat{D}_1)\partial_t \hat{z} = \partial_x \hat{z} + (D-\hat{D}_1)\Phi((D-\hat{D}_1)x, \hat{D}_1)A_1\tilde{\zeta}(0,t)$$

Then, consider $x \in [\hat{D}_1/D, 1]$. From (16), it follows that

$$\partial_t \hat{v}(x,t) = \partial_t \hat{z}\left(\frac{Dx-\hat{D}_1}{D-\hat{D}_1}, t\right), \partial_x \hat{v}(x,t) = \tfrac{D}{D-\hat{D}_1}\partial_x \hat{z}\left(\frac{Dx-\hat{D}_1}{D-\hat{D}_1}, t\right)$$

Plugging the two previous equations, one obtains (17). Further, consider $x \in [0, \hat{D}_1/D)$. We have

$$\partial_t \hat{v}(x,t) = \partial_t \hat{\zeta}\left(\frac{D}{\hat{D}_1}x, t\right), \quad \partial_x \hat{v}(x,t) = \frac{D}{\hat{D}_1}\partial_x \hat{\zeta}\left(\frac{D}{\hat{D}_1}x, t\right)$$

Using the dynamics of $\hat{\zeta}$ given in Lemma 1, one obtains $D\partial_t \hat{v} = \partial_x \hat{v}$ which is indeed (17) as $\Phi(Dx - \hat{D}_1, \hat{D}_1) = 0$ for $x \in [0, \hat{D}_1, D)$. Finally, from (16), $\hat{v}(0,t) = \hat{\zeta}(0,t)$. ∎

This lemma simply states that the normalized result of two consecutive transport equations is a transport equation with concatenated source terms and the speed of which is the sum of the original two. In other words, plugging together $\hat{\zeta}$ and $\hat{z}$ with respective speeds of propagation $\hat{D}_1$ and $D - \hat{D}_1$ and with similar boundary condition $\hat{\zeta}(1,t) = X(t) = \hat{v}(0,t)$, one obtains one unique transport equation with speed of propagation D.

Backstepping transformation and target system Consider the backstepping transformation of the distributed actuator u

$$\begin{aligned}\hat{w}(x,t) =& u(x,t) - K_1\hat{v}(x,t) - K_0\Phi(Dx, \hat{D}_1)X(t)\\ &- K_0\hat{D}_1\int_0^1 \Phi(Dx - \hat{D}_1 y, \hat{D}_1)A_1\hat{\zeta}(y,t)dy\\ &- K_0 D\int_0^x \Phi(D(x-y), \hat{D}_1)Bu(y,t)dy \qquad (18)\end{aligned}$$

in which the distributed variable $\hat{v}$ is introduced in (16). This backstepping transformation is defined in order to fulfill the boundary condition $\hat{w}(1,t) = 0$, which is consistent with (9). This particular property enables then to introduce negative bounding terms in the Lyapunov analysis which are used to prove stability in the following section.

Lemma 3. The backstepping transformation (18) together with the control law (9) transform the plant (8) into

$$\begin{cases}\dot{X}(t) = (A_0 + BK_0)X(t)\\ \qquad + (A_1 + BK_1)\zeta(0,t) + B[\hat{w}(0,t) - K_1\tilde{\zeta}(0,t)]\\ \begin{cases} D_1\partial_t\zeta = \partial_x\zeta\\ \zeta(1,t) = X(t)\end{cases} \qquad \begin{cases} D\partial_t\hat{w} = \partial_x\hat{w} - g(x)\tilde{\zeta}(0,t)\\ \hat{w}(1,t) = 0\end{cases}\\ \begin{cases} D_1\partial_t\tilde{\zeta} = \partial_x\tilde{\zeta} - \tilde{D}_1 h(x,t)\\ \tilde{\zeta}(1,t) = 0\end{cases}\end{cases}$$

in which $g(x) = D[K_0\Phi(Dx, \hat{D}_1) + K_1\Phi(Dx - \hat{D}_1, \hat{D}_1)]A_1$ and $h(x,t) = \frac{\partial_x\hat{\zeta}}{\hat{D}_1}$. Further, the spatial-derivative of the distributed state estimate (12) satisfies the following dynamics

$$\begin{cases}\hat{D}_1\partial_{xt}\hat{\zeta} = \partial_{xx}\hat{\zeta}\\ \partial_x\hat{\zeta}(1,t) = \hat{D}_1\big((A_0 + BK_0)X(t) + (A_1 + BK_1)\zeta(0,t)\\ \qquad + B[\hat{w}(0,t) - K_1\tilde{\zeta}(0,t)]\big)\end{cases}$$

Proof: First, using (18), the initial condition of $\Phi(\cdot, \hat{D}_1)$ and (17), one obtains that

$$\begin{aligned}u(0,t) =& \hat{w}(0,t) + K_1\hat{\zeta}(0,t) + K_0X(t)\\ =& \hat{w}(0,t) - K_1\tilde{\zeta}(0,t) + K_1\zeta(0,t) + K_0X(t)\end{aligned}$$

and therefore the first ODE follows. Using Lemma 1, the only transport PDEs that remain to study are the one governing the backstepping transformation $\hat{w}$ and the spatial-derivative of the distributed state estimate $\partial_x\hat{\zeta}$. With this aim in view, define the intermediate variable

$$\begin{aligned}v_0(x,t) =& \Phi(Dx, \hat{D}_1)X(t) + \hat{D}_1\int_0^1 \Phi(Dx - \hat{D}_1 y, \hat{D}_1)A_1\hat{\zeta}(y,t)dy\\ &+ D\int_0^x \Phi(D(x-y), \hat{D}_1)Bu(y,t)dy\end{aligned}$$

in terms of which the backstepping transformation can be expressed as $\hat{w}(x,t) = u(x,t) - K_1\hat{v}(x,t) - K_0 v_0(x,t)$. Following lines similar to those previously used for Lemma 2, one obtains

$$D\partial_t v_0 = \partial_x v_0 + D\Phi(Dx, \hat{D}_1)A_1\tilde{\zeta}(0,t) \quad v_0(0,t) = X(t)$$

Using (8) and (17), it follows that $D\partial_t\hat{w} = \partial_x\hat{w} - g(x)\tilde{\zeta}(0,t)$ which gives the desired result. The boundary condition $\hat{w}(1,t) = 0$ can be obtained from (9) with changes of variable.

Finally, the dynamics of $\partial_x\hat{\zeta}$ can be obtained by taking a spatial-derivative of the dynamics of $\hat{\zeta}$ which is given in the proof of Lemma 1. The boundary condition can also be obtained from this dynamics for $x = 1$, as, following (12), $\partial_x\hat{\zeta}(1,t) = \hat{D}_1\dot{X}(t)$ which is given by the first ODE in the statement of Lemma 3. ∎

Lyapunov analysis We now rely on the elements gathered in Appendix to define the following Lyapunov-Krasovskii functional candidate

$$\begin{aligned}V(t) =& W(t) + b_0 D_1\int_0^1 (1+x)|\tilde{\zeta}(x,t)|^2 dx \qquad (19)\\ &+ b_1 D\int_0^1 (1+x)\hat{w}(x,t)^2 dx + b_2\hat{D}_1\int_0^1 (1+x)|\partial_x\hat{\zeta}(x,t)|^2 dx\end{aligned}$$

in which the functional W is given in (A.1). Taking a time-derivative of (19) and using Lemma 7 given in Appendix with $\varepsilon(t) = B[\hat{w}(0,t) - K_1\tilde{\zeta}(0,t)]$, one obtains

$$\begin{aligned}\dot{V}(t) =& -X(t)^T W_0 X(t) - \zeta(0,t)^T W_1\zeta(0,t)\\ &- D_1\int_0^1 \zeta(x,t)^T W_2\zeta(x,t)dx + 2X^T Q(0)B[\hat{w}(0,t) - K_1\tilde{\zeta}(0,t)]\\ &+ 2D_1(\hat{w}(0,t) - K_1\tilde{\zeta}(0,t))B^T\int_0^1 Q(-D_1 x)(A_1 + BK_1)\zeta(x,t)dx\\ &- b_0|\tilde{\zeta}(0,t)|^2 - b_0\left\|\tilde{\zeta}(t)\right\|^2 - 2b_0\tilde{D}_1\int_0^1 (1+x)\tilde{\zeta}(x,t)^T h(x,t)dx\\ &- b_1\hat{w}(0,t)^2 - b_1\|\hat{w}(t)\|^2 - 2b_1\int_0^1 (1+x)\hat{w}(x,t)g(x)\tilde{\zeta}(0,t)dx\\ &+ b_2(2|\partial_x\hat{\zeta}(1,t)|^2 - |\partial_x\hat{\zeta}(0,t)|^2 - \left\|\partial_x\hat{\zeta}(t)\right\|^2)\end{aligned}$$

in which we have used the dynamics given in Lemma 3, suitable integrations by parts and finally changes of variable to express some integrals in terms of ζ. From the expressions of g, h and $\partial_x\hat{\zeta}(1,t)$ given in Lemma 3, one obtains, applying Cauchy-Schwartz's and Young's inequalities, the existence of a constant $M > 0$ such that

$$\left|2\tilde{D}_1\int_0^1 (1+x)\tilde{\zeta}(x,t)^T h(x,t)dx\right| \leq M\tilde{D}_1^2\left\|\partial_x\hat{\zeta}(t)\right\|^2 + \frac{\left\|\tilde{\zeta}(t)\right\|^2}{2}$$

$$\left|2\int_0^1 (1+x)\hat{w}(x,t)g(x)\tilde{\zeta}(0,t)dx\right| \leq M|\tilde{\zeta}(0,t)|^2 + \|\hat{w}(t)\|^2/2$$

$$2|\partial_x\hat{\zeta}(1,t)|^2 \leq M(|X(t)|^2 + |\zeta(0,t)|^2 + |\tilde{\zeta}(0,t)|^2 + \hat{w}(0,t)^2)$$

Therefore, applying Young's inequality and defining

$$M_0 = \frac{4|Q(0)B|^2}{\lambda_m(W_0)} + \frac{4D_1}{\lambda_m(W_2)}\left|B^T \max_{x\in[0,1]} Q(-D_1x)(A_1+BK_1)\right|^2$$

it follows

$$\begin{aligned}\dot V(t) \le &-\left(\frac{\lambda_m(W_0)}{2} - b_2M\right)|X(t)|^2 - \frac{\lambda_m(W_2)D_1}{2}\|\zeta(t)\|^2\\ &-(\lambda_m(W_1)-b_2M)|\zeta(0,t)|^2 - b_2\left\|\partial_x\hat\zeta(t)\right\| - \frac{b_0}{2}\left\|\tilde\zeta(t)\right\|^2\\ &-\left(b_0 - M_0|K_1|^2-(b_1+b_2)M\right)|\tilde\zeta(0,t)|^2 - \frac{b_1}{2}\|\hat w(t)\|^2\\ &-\left(b_1-M_0-b_2M\right)\hat w(0,t)^2 + b_0\tilde D_1^2M\left\|\partial_x\hat\zeta(t)\right\|^2\end{aligned}$$

Consequently, choosing $0<b_2<\min\left\{\frac{\lambda_m(W_0)}{2M},\frac{\lambda_m(W_1)}{M}\right\}$, $b_1 > M_0+b_2M$, $b_0 > M_0|K_1|^2+(b_1+b_2)M$, and defining $\eta = \min\left\{\frac{\lambda_m(W_0)}{2}-b_2M, \frac{\lambda_m(W_2)D_1}{2}, b_2, \frac{b_0}{2}, \frac{b_1}{2}\right\}$, one gets

$$\begin{aligned}\dot V(t) \le &-\left(\eta - b_0\tilde D_1^2M\right)(|X(t)|^2 + \|\zeta(t)\|^2 + \left\|\tilde\zeta(t)\right\|^2\\ &+\left\|\partial_x\hat\zeta(t)\right\|^2 + \|\hat w(t)\|^2)\end{aligned}$$

Therefore, if $|\tilde D_1| < \sqrt{\frac{\eta}{b_0M}} = \delta^*$, using (A.3), there exists $\eta_0 > 0$ such that $\dot V(t) \le -\eta_0 V(t)$ and consequently $V(t) \le V(0)e^{-\eta_0 t}$ for $t\ge 0$. Now it remains to relate Γ and V to formulate a similar property for Γ.

Equivalence between the two functionals Γ and V Considering the transformation (18) and its inverse (the expression of which is not given here due to space limitation but can be obtained similarly to the one given in the proof of Lemma 8 in Appendix), one can obtain the existence of positive constants r_1, r_2, r_3, s_1, s_2 and s_3 such that

$$\begin{aligned}\|u(t)\|^2 \le& r_1|X(t)|^2 + r_2\left\|\hat\zeta(t)\right\|^2 + r_3\|\hat w(t)\|^2\\ \|\hat w(t)\|^2 \le& s_1|X(t)|^2 + s_2\left\|\hat\zeta(t)\right\|^2 + s_3\|u(t)\|^2\end{aligned}$$

Using these two inequalities together with (A.3), one can obtain, with straightforward inequalities and changes of variable, the existence of $\beta_1,\beta_2>0$ such that

$$\beta_1\Gamma(t) \le V(t) \le \beta_2\Gamma(t)$$

Consequently, it follows that

$$\Gamma(t) \le \frac{V(t)}{\beta_1} \le \frac{V(0)}{\beta_1}e^{-\eta_0 t} \le \frac{\beta_2}{\beta_1}\Gamma(0)e^{-\eta_0 t},\ t\ge 0$$

This concludes the proof of Theorem 1.

5. ROBUSTNESS TO INPUT DELAY MISMATCH

In this section, we now consider that $D\in[\underline{D},\bar D]$ is uncertain, while the state delay D_1 is perfectly known. Correspondingly, we define a constant delay estimate $\hat D$ and follow steps similar to those used in the previous section.

5.1 Control design

Following the certainty equivalence principle applied to the nominal controller (7), we choose the control law

$$\begin{aligned}U(t) =& [K_0\Phi(\hat D, D_1) + K_1\Phi(\hat D - D_1, D_1)]X(t)\\ &+\int_{t-D_1}^t [K_0\Phi(t+\hat D - D_1 - s, D_1)\\ &+K_1\Phi(t+\hat D - 2D_1 - s, D_1)]A_1X(s)ds\\ &+\int_{t-\hat D}^t K_0\Phi(t-s,D_1)BU(s)ds\\ &+\int_{t-\hat D}^{t-D_1} K_1\Phi(t-D_1-s,D_1)BU(s)ds \qquad (20)\end{aligned}$$

Theorem 2. Consider the closed-loop system consisting of the plant (1) and the control law (20). Define the functional

$$\begin{aligned}\Gamma(t) =& |X(t)|^2 + \int_{t-2D_1}^t |X(s)|^2ds\\ &+\int_{t-\max\{D,\hat D\}-D_1}^t U(s)^2ds + \int_{t-\hat D}^t \dot U(s)^2ds\end{aligned}$$

There exists $\delta^*>0$ such that, if $|\tilde D| < \delta^*$, there exists $R>0$ and $\rho>0$ such that

$$\Gamma(t) \le R\Gamma(0)e^{-\rho t},\ t\ge 0 \qquad (21)$$

Note that the functional Γ introduced in Theorem 2 is different from the one given in Theorem 1 in the sense that it also involves the $\mathscr{L}_2$-norm of the control map. This is due to the fact that the source terms appearing in the dynamics involves a derivative of the input because of the input history estimation.

5.2 Proof of Theorem 2

Distributed estimates We introduce the following distributed input estimate with the corresponding distributed input error

$$\hat u(x,t) = U(t+\hat D(x-1)) \qquad (22)$$

$$\tilde u(x,t) = u(x,t) - \hat u(x,t) \qquad (23)$$

Lemma 4. Consider the distributed variables defined in (22)–(23). The plant (8) can then be reformulated as

$$\begin{cases}\dot X(t) = A_0X + A_1\zeta(0,t) + B[\tilde u(0,t) + \hat u(0,t)]\\ \begin{cases}D_1\partial_t\zeta = \partial_x\zeta\\ \zeta(1,t) = X(t)\end{cases} \quad \begin{cases}D\partial_t\tilde u = \partial_x\tilde u - \tilde D h_0(x,t)\\ \tilde u(1,t) = 0\end{cases}\\ \qquad\qquad\qquad\qquad \begin{cases}\hat D\partial_t\hat u = \partial_x\hat u\\ \hat u(1,t) = U(t)\end{cases}\end{cases} \qquad (24)$$

in which $\tilde D = D - \hat D$ and $h_0(x,t) = \partial_x\hat u(x,t)/\hat D$.

Proof: From (23), it follows that $u(0,t) = \tilde u(0,t) + \hat u(0,t)$ which gives the plant ODE. Second, taking spatial- and time-derivatives of (22), one obtains that $\hat D\partial_t\hat u = \partial_x\hat u$. Therefore, using the equation governing u in (8), one gets

$$D\partial_t\tilde u = \partial_x u - \frac{D}{\hat D}\partial_x\hat u = \partial_x\tilde u - \frac{\tilde D}{\hat D}\partial_x\hat u$$

Using finally that $\hat u(1,t) = u(1,t) = U(t)$, the boundary condition $\tilde u(1,t) = 0$ follows. ∎

Lemma 5. Define the distributed variables

$$\begin{aligned}\hat z(x,t) =& \Phi((\hat D - D_1)x, D_1)X(t)\\ &+D_1\int_0^1 \Phi((\hat D - D_1)x - D_1y, D_1)A_1\zeta(y,t)dy\\ &+\hat D\int_0^{\frac{\hat D - D_1}{\hat D}x} \Phi((\hat D - D_1)x - \hat Dy, D_1)B\hat u(y,t)dy \qquad (25)\end{aligned}$$

$$\hat v(x,t) = \begin{cases}\zeta\left(\frac{\hat D}{D_1}x, t\right) & \text{for } x\in[0, D_1/\hat D)\\ \hat z\left(\frac{\hat Dx - D_1}{\hat D - D_1}, t\right) & \text{for } x\in[D_1/\hat D, 1]\end{cases} \qquad (26)$$

The the following dynamics is satisfied, for $x\in[0,1]$ and $t\ge 0$,

$$\begin{cases}\hat D\partial_t\hat v = \partial_x\hat v + \hat D\Phi(\hat Dx - D_1, D_1)B\tilde u(0,t)\\ \hat v(0,t) = \zeta(0,t)\end{cases} \qquad (27)$$

Proof: The proof follows steps similar to those used for Lemma 2. First, one can observe that (27) makes sense because $\hat{v}(D_1/\hat{D}) = \zeta(1,t) = \hat{z}(0,t) = X(t)$. Now, taking spatial- and time-derivatives of (25), as previously, one can get $(\hat{D}-D_1)\partial_t\hat{z} = \partial_x\hat{z} + (\hat{D}-D_1)\Phi((\hat{D}-D_1)x, D_1)B\tilde{u}(0,t)$ with $\hat{z}(0,t) = X(t)$. Taking spatial and time-derivatives of (16) for $x \in [D_1/\hat{D}, 1]$ and $x \in [0, D_1/\hat{D})$ respectively, using this last dynamics and the one of ζ given in Lemma 4, one concludes. ∎

5.3 Backstepping transformation and target system

We now define the following backstepping transformation

$$\begin{aligned}\hat{w}(x,t) =& \hat{u}(x,t) - K_1\hat{v}(x,t) - K_0\Phi(\hat{D}x, D_1)X(t) \\ &- K_0 D_1 \int_0^1 \Phi(\hat{D}x - D_1 y, D_1)A_1\zeta(y,t)dy \\ &- K_0\hat{D}\int_0^x \Phi(\hat{D}(x-y), D_1)B\hat{u}(y,t)dy \end{aligned} \tag{28}$$

Lemma 6. The backstepping transformation (28) together with the control law (20) transform the plant (8) into

$$\begin{cases} \dot{X}(t) = (A_0 + BK_0)X(t) + (A_1 + BK_1)\zeta(0,t) \\ \qquad\qquad + B[\tilde{u}(0,t) + \hat{w}(0,t)] \\ \begin{cases} D_1\zeta_t = \zeta_x \\ \zeta(1,t) = X(t) \end{cases} \quad \begin{cases} D\partial_t\tilde{u} = \partial_x\tilde{u} - \tilde{D}h_0(x,t) \\ \tilde{u}(1,t) = 0 \end{cases} \\ \qquad\qquad\qquad\quad \begin{cases} \hat{D}\partial_t\hat{w} = \partial_x\hat{w} - g_0(x)\tilde{u}(0,t) \\ \hat{w}(1,t) = 0 \end{cases} \end{cases}$$

in which $g_0(x) = \hat{D}[K_0\Phi(\hat{D}x, D_1) + K_1\Phi(\hat{D}x - D_1, D_1)]B$ and $h_0(x,t) = \frac{\partial_x\hat{u}}{\hat{D}}$ which can be reformulated using Lemma 8 in Appendix. Further, the spatial-derivatives of the backstepping transformation and of the distributed state satisfy the following dynamics

$$\begin{cases} \hat{D}\partial_{xt}\hat{w} = \partial_{xx}\hat{w} - \partial_x g_0(x)\tilde{u}(0,t) \\ \partial_x\hat{w}(1,t) = g_0(1)\tilde{u}(0,t) \end{cases}$$

$$\begin{cases} D_1\partial_{xt}\zeta = \partial_{xx}\zeta \\ \partial_x\zeta(1,t) = D_1((A_0 + BK_0)X(t) + (A_1 + BK_1)\zeta(0,t) \\ \qquad\qquad + B[\tilde{u}(0,t) + \hat{w}(0,t)]) \end{cases}$$

Proof: Define the intermediate variable

$$\begin{aligned} v_0(x,t) =& \Phi(\hat{D}x, D_1)X(t) + D_1\int_0^1 \Phi(\hat{D}x - D_1 y, D_1)A_1\zeta(y,t)dy \\ &+ \hat{D}\int_0^x \Phi(\hat{D}(x-y), D_1)B\hat{u}(y,t)dy \end{aligned}$$

in terms of which the backstepping transformation can be expressed as $\hat{w}(x,t) = \hat{u}(x,t) - K_1\hat{v}(x,t) - K_0 v_0(x,t)$. Following lines similar to those previously used in Lemma 2, one obtains that $\hat{D}\partial_t v_0 = \partial_x v_0 + \hat{D}\Phi(\hat{D}x, D_1)B\tilde{u}(0,t)$ with $v_0(0,t) = X(t)$. Therefore, using Lemma (5) and the dynamics governing $\hat{u}$ in (24), one gets that $\hat{D}\partial_t\hat{w} = \partial_x\hat{w} - g_0(x)\tilde{u}(0,t)$. The boundary condition $\hat{w}(1,t) = 0$ can be obtained from (20) with suitable changes of variable.

Finally, the dynamics of $\partial_x\hat{w}$ and $\partial_x\zeta$ can be obtained by taking a spatial-derivative of the dynamics of $\hat{w}$ and of ζ respectively. The boundary conditions follows also from these dynamics noticing that $\partial_t\hat{w}(1,t) = 0$ as $\hat{w}(1,t) = 0$ for all time. ∎

Lyapunov analysis We now consider the following Lyapunov-Krasovskii functional candidate

$$\begin{aligned} V(t) =& W(t) + b_0 D_1 \int_0^1 (1+x)|\partial_x\zeta(x,t)|^2 dx \\ &+ b_1 D\int_0^1 (1+x)\tilde{u}(x,t)^2 dx + b_2\hat{D}\int_0^1 (1+x)\hat{w}(x,t)^2 dx \\ &+ b_2\hat{D}\int_0^1 (1+x)(\partial_x\hat{w}(x,t))^2 dx \end{aligned} \tag{29}$$

in which W is defined in (A.1) in Appendix. Taking a time-derivative of (29), using the dynamics given in Lemma 6 and Lemma (7) given in Appendix with $\varepsilon(t) = B[\hat{w}(0,t) + \tilde{u}(0,t)]$, one obtains

$$\begin{aligned} \dot{V}(t) =& -X(t)^T W_0 X(t) - \zeta(0,t)^T W_1\zeta(0,t) \\ &- D_1\int_0^1 \zeta(x,t)^T W_2\zeta(x,t)dx + 2X^T Q(0)B[\hat{w}(0,t) + \tilde{u}(0,t)] \\ &+ 2D_1(\hat{w}(0,t) + \tilde{u}(0,t))B^T\int_0^1 Q(-D_1 x)(A_1 + BK_1)\zeta(x,t)dx \\ &+ b_0(2|\partial_x\zeta(1,t)|^2 - |\partial_x\zeta(0,t)|^2 - \|\partial_x\zeta(t)\|^2) - b_1\tilde{u}(0,t)^2 \\ &- b_1\|\tilde{u}(t)\|^2 - 2b_1\tilde{D}\int_0^1 (1+x)h_0(x,t)\tilde{u}(x,t)dx - b_2\hat{w}(0,t)^2 \\ &- 2b_2\int_0^1 (1+x)[\hat{w}(x,t)g_0(x) + \partial_x\hat{w}(x,t)\partial_x g_0(x)]\tilde{u}(0,t)dx \\ &- b_2\|\hat{w}(t)\|^2 + b_2(2(\partial_x\hat{w}(1,t))^2 - (\partial_x\hat{w}(0,t))^2 - \|\partial_x\hat{w}(t)\|^2) \end{aligned}$$

in which we have used suitable integrations by parts and finally changes of variable to express some integrals in terms of ζ. Using the expressions of h and $\partial_x\zeta(1,t)$ provided in Lemma 6 and Lemma 8 given in Appendix, one obtains, applying Cauchy-Schwartz's and Young's inequalities, the existence of a constant $M > 0$ such that

$$\begin{aligned} &\left|2\tilde{D}\int_0^1 (1+x)h_0(x,t)\tilde{u}(x,t)dx\right| \le \frac{\|\tilde{u}(t)\|^2}{2} + M\tilde{D}^2 \\ &\quad\times\left(\|\partial_x\hat{w}(t)\|^2 + \|\partial_x\zeta(t)\|^2 + |X(t)|^2 + \|\zeta(t)\|^2 + \|\hat{w}(t)\|^2\right) \\ &2|\partial_x\zeta(1,t)|^2 \le M(|X(t)|^2 + |\zeta(0,t)|^2 + \tilde{u}(0,t)^2 + \hat{w}(0,t)^2) \\ &\left|2\int_0^1 (1+x)[\hat{w}(x,t)g_0(x) + \partial_x\hat{w}(x,t)\partial_x g_0(x)]\tilde{u}(0,t)dx\right| \\ &\qquad\qquad \le \|\hat{w}(t)\|^2/2 + \|\partial_x\hat{w}(t)\|^2/2 + M\tilde{u}(0,t)^2 \\ &2(\partial_x\hat{w}(1,t))^2 \le M\tilde{u}(0,t)^2 \end{aligned}$$

Therefore, applying Young's inequality, it follows

$$\begin{aligned} \dot{V}(t) \le& -\left(\frac{\lambda_m(W_0)}{2} - b_0 M\right)|X(t)|^2 - \frac{\lambda_m(W_2)D_1}{2}\|\zeta(t)\|^2 \\ &- b_0\|\partial_x\zeta(t)\| - \frac{b_1}{2}\|\tilde{u}(t)\|^2 - \frac{b_2}{2}\|\hat{w}(t)\|^2 - \frac{b_2}{2}\|\partial_x\hat{w}(t)\|^2 \\ &- (\lambda_m(W_1) - b_0 M)|\zeta(0,t)| + b_1\tilde{D}^2 M\big(\|\partial_x\hat{w}(t)\|^2 \\ &+ \|\partial_x\zeta(t)\|^2 + |X(t)|^2 + \|\zeta(t)\|^2 + \|\hat{w}(t)\|^2\big) \\ &- (b_1 - M_0 - (b_0 + 2b_2)M)\tilde{u}(0,t)^2 - (b_2 - M_0 - b_0 M)\hat{w}(0,t)^2 \end{aligned}$$

in which M_0 has already be defined in Section (4.2). Consequently, choosing $0 < b_0 < \min\left\{\frac{\lambda_m(W_0)}{2M}, \frac{\lambda_m(W_1)}{M}\right\}$, $b_2 > M_0 + b_0 M$, $b_1 > M_0 + (b_0 + 2b_2)M$ and defining $\eta = \min\left\{\frac{\lambda_m(W_0)}{2} - b_0 M, \frac{\lambda_m(W_2)D_1}{2}, b_0, \frac{b_1}{2}, \frac{b_2}{2}\right\}$, one gets

$$\begin{aligned} \dot{V}(t) \le& -\left(\eta - b_1\tilde{D}^2 M\right)(|X(t)|^2 + \|\zeta(t)\|^2 + \|\partial_x\zeta(t)\|^2 \\ &+ \|\tilde{u}(t)\|^2 + \|\hat{w}(t)\|^2 + \|\partial_x\hat{w}(t)\|^2) \end{aligned}$$

Therefore, if $|\tilde{D}| < \sqrt{\frac{\eta}{b_1 M}} = \delta^*$, using (A.3) and straightforward inequalities and change of variables, there exists $\eta_0 > 0$ such that $\dot{V}(t) \le -\eta_0 V(t)$ and therefore $V(t) \le V(0)e^{-\eta_0 t}$ for $t \ge 0$. It now remains to show the same result in terms of the functional Γ.

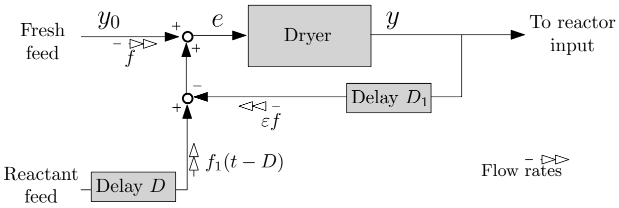

Fig. 1. Schematic view of the dryer system under consideration. The double arrows represent the (measured) flow rates of the system. The dryer treats a fresh feed to provide to a reactor a water-free solution, with Reactant R concentration y. The system involves a recycle loop. Due to transportation through the pipes, both input and state delays occur.

Equivalence between the two functionals Γ *and* V Considering the transformation (28) and its inverse given in the proof of Lemma 8 in Appendix, one can obtain the existence of positive constants r_1, r_2, r_3, s_1, s_2 and s_3 such that

$$\|\hat{u}(t)\|^2 \leq r_1|X(t)|^2 + r_2\|\zeta(t)\|^2 + r_3\|\hat{w}(t)\|^2$$
$$\|\hat{w}(t)\|^2 \leq s_1|X(t)|^2 + s_2\|\zeta(t)\|^2 + s_3\|\hat{u}(t)\|^2$$

and similar ones for the spatial-derivatives $\hat{u}_x$ and $\hat{w}_x$. Using these two inequalities together with (A.3), one can obtain, with straightforward inequalities and changes of variable, the existence of $\beta_1, \beta_2 > 0$ such that $\beta_1\Gamma(t) \leq V(t) \leq \beta_2\Gamma(t)$ for $t \geq 0$ which gives (21).

6. SIMULATION EXAMPLE

In this section, to illustrate the merits of the obtained results, we consider an example from the process industry, which can be found in petrochemical plants [3]. This drying process aims at eliminating the water from a fresh feed to be treated in a polymerization reactor, in which an hydrophobic reaction occurs. A second main objective of the process is to control the concentration in one of the reactant R of the feed entering the reactor. To fulfill this aim, an extra amount of reactant R can be provided at the input of the dryer. This is the control variable of the process.

The system under consideration is pictured in Fig. 1. To improve the drying efficiency, a portion of the output defined by ε is recycled. This recycle loop together with the pipes network used to provide the desired amount of reactant R involve flow transportation which result into two transport delays, one bearing on the system state (D_1) and the second one on the input (D). Note that this intricate configuration is a consequence of the network architecture, historic design choices and hardware upgrades, which are now suffered. This explains the resulting unwanted transport delays.

For the sake of simplicity of the exposure, we consider here that the fresh feed concentration y_0 and the various mass flow rates are constant. The dryer dynamics can be approximated by a first-order stable equation with unitary static gain [4]. We

[3] We wish to thank A. Le Walle from Total for suggesting this study.

[4] A more realistic model would be a delayed first order equation with a time-varying gain. Even if we do not consider this delay here, for the sake of clarity, the control design could be straightforwardly modified to account for it. However, gain variations is a much more challenging topic which cannot be handled by the current proposed methodology and would require additional integral controller. This is a direction of future work.

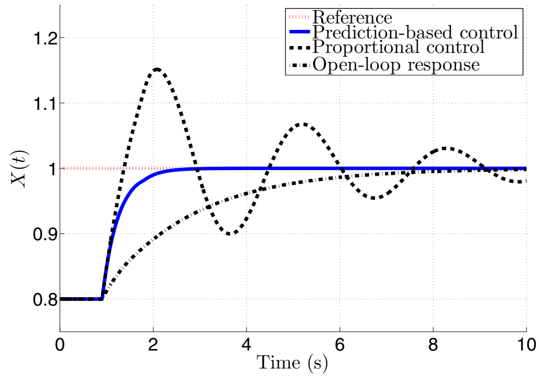

Fig. 2. Simulation results, starting from the equilibrium $(X_0, U_0) = (0.8, 0.8f)$ and with three different controllers: the proposed prediction-based control law (plain) with gains $(K_0, K_1) = (-2A_0/B, -A_1/B)$, a proportional one (dashed) with gain $K = -2A_0/B$ and the open-loop response (dashed dotted). The parameters values are $f = 1, \tau = 1, \varepsilon = 0.5, y_0 = 0.8, D_1 = 0.7$ and $D = 0.9$.

further assume that the transport delays are constant [5] and that, due to the magnitude of the flow rate and pipes length, $D_1 < D$. Under these assumptions, the dryer inlet concentration is

$$e(t) = \frac{fy_0 + \varepsilon fy(t-D_1) + f_1(t-D)}{f(1+\varepsilon) + f_1(t-D)}$$

Therefore, using the fact that $f_1 << 1$ due to the scale of the concentrations and volumes at stake, the dynamics can be approximated as

$$\tau\dot{y}(t) = -y(t) + \frac{fy_0 + \varepsilon fy(t-D_1) + f_1(t-D)}{f(1+\varepsilon)}$$

which complies with (1), defining $X = y$, $U = fy_0 + f_1$, $A_0 = -\frac{1}{\tau}$, $A_1 = \frac{\varepsilon}{\tau(1+\varepsilon)}$ and $B = \frac{1}{\tau f(1+\varepsilon)}$.

One can check that, for the considered parameters values, the open-loop dynamics is exponentially stable. However, to improve transient performances, we consider the set of feedback gain $\mathbb{R}_- \times \{-A_1/B\}$. This set satisfies Assumption 1, as (2) would be $\dot{X} = -\left(\frac{1}{\tau} + k\right)X(t)$ which is exponentially stable for $k \geq 0$. Finally, the control reference corresponding to a state equilibrium X^r is $U^r = fX^r$.

To implement the control laws (9) and (20), we solve (3) numerically on the time interval $[0, D]$ by inductive forward Euler approximation of (3) on intervals of length D_1 (starting with $\Phi_t = A_0\Phi(t)$ for $t \in [0, D_1]$ using the initial condition of the transition matrix). The integrals involved in the two controllers equations (9) and (20) are computed using trapezoidal approximations.

Fig. 2 pictures the simulation response obtained with the proposed controller for $k = -2/\tau$ and with the true delays. For the sake of comparison, the closed-loop response obtained with a proportional controller employing the same feedback gain and the open-loop response are also provided. One can observe that the proposed controller achieves exponential convergence and considerably increases the response time compared to the open-loop response. This is not achieved by a simple proportional controller which exhibits substantial oscillations. Decreasing

[5] In practice, these delays vary with the flow rates according to an integral equation.

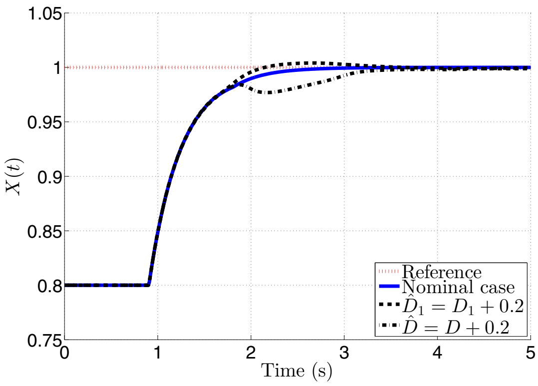

Fig. 3. Simulation results, starting from the equilibrium $(X_0, U_0) = (0.8, 0.8(d(1+\varepsilon)-\varepsilon))$, with the proposed prediction-based control law with gains $(K_0, K_1) = (-2A_0, -A_1)$ for three different sets of delays: the true ones (plain), an overestimated state delay (dashed) and an overestimated input delay (dashed dotted). The parameters values are $f = 1, \tau = 1, \varepsilon = 0.5$ $y_0 = 0.8$, $D_1 = 0.7$ and $D = 0.9$.

the magnitude of the feedback gain would in all likelihood suppress these oscillations but would also worsen the time response of the system. In any case, the time response obtained with the proportional controller would be more important than the one obtained with the proposed prediction-based controller. This is the main advantage of the proposed technique.

Fig. 3 aims at illustrating Theorems 1 and 2. The closed-loop response obtained employing respectively an erroneous state delay $\hat{D}_1 = D_1 + 0.2$ and an erroneous input delay $\hat{D} = D + 0.2$ are compared to the nominal response. One can observe that transient performances are decreased compared to the nominal case, due to delay estimation errors, but that convergence is still achieved with an honorable response time. The effect of the delay errors appear at approximately $t \approx 1.8 = 2D$, i.e. roughly after a time horizon corresponding to two input delays. This is due to the fact that the effect of the errors perpetrated while computing the transition matrix does not arise in the control law before this time, as the first values of the transition matrix are correct (similar initial conditions).

7. CONCLUSION

In this paper, we addressed prediction-based design for systems with both state and input delays and proved robustness of the controller provided that delays mismatches are small enough. Our approach is grounded on the linking of two recently proposed infinite-dimensional techniques: a Complete-Type Lyapunov functional, which enables state delay systems stability analysis, and tools from the field of Partial Differential Equations, reformulating the delays as transport equations and introducing a tailored backstepping transformation. We claim that these elements can be straightforwardly extended to design delay-adaptive control. This is a direction of future work.

Appendix A. COMPLETE-TYPE LYAPUNOV-KRASOVSKII FUNCTIONALS

Consider the following functional from Kharitonov [2013b]

$$W(t) = X(t)^T Q(0) X(t) + \int_{t-D_1}^{t} X(s_1)(A_1+BK_1)^T \times \int_{t-D_1}^{t} Q(s_1-s_2)(A_1+BK_1)X(s_2)ds_1 ds_2 + 2X(t)^T \int_{t-D_1}^{t} Q(t-D_1-s)(A_1+BK_1)X(s)ds + \int_{t-D_1}^{t} X(s)^T [W_1 + (s-t+D_1)W_2] X(s) ds \quad \text{(A.1)}$$

in which W_0, W_1 and W_2 are given positive definite matrices and

$$Q(t) = \int_0^{\infty} \tilde{\Phi}(s)(W_0 + W_1 + D_1 W_2)\tilde{\Phi}(s+t)ds$$

with $\tilde{\Phi}$ the characteristic matrix of (2).

Proposition 3. The time-derivative of the functional (A.1) along any trajectory of (2) satisfies

$$\dot{W}(t) = -X(t)^T W_0 X(t) - X(t-D_1)^T W_1 X(t-D_1) - \int_{t-D_1}^{t} X(s)^T W_2 X(s) ds \quad \text{(A.2)}$$

Further, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\alpha_1 |X(t)|^2 \leq W(t) \leq \alpha_2 \left(|X(t)|^2 + \int_{t-D_1}^{t} |X(t)|^2 ds \right) \quad \text{(A.3)}$$

Lemma 7. The time-derivative of the functional (A.1) along any trajectory of the plant

$$\dot{X}(t) = (A_0 + BK_0)X(t) + (A_1 + BK_1)X(t-D_1) + \varepsilon(t) \quad \text{(A.4)}$$

satisfies

$$\dot{W}(t) = -X(t)^T W_0 X(t) - X(t-D_1)^T W_1 X(t-D_1) - \int_{t-D_1}^{t} X(s)^T W_2 X(s) ds + 2X(t)^T Q(0)\varepsilon(t) + 2\varepsilon(t)^T \int_{t-D_1}^{t} Q(t-D_1-s)(A_1+BK_1)X(s)ds$$

Elements of proof: One can observe that only the first and the third terms in (A.1) involve system state values that explicitly depend on the time-parameter t. Therefore, while taking a time-derivative of (A.1), the differential equation of X only matters to obtain a time-derivative of the first and third terms. Comparing (2) and (A.4). one obtains for the first term, for example,

$$\frac{d}{dt}[X(t)^T Q(0) X(t)]_{|(A.4)} = \frac{d}{dt}[X(t)^T Q(0) X(t)]_{|(2)} + 2\varepsilon(t)^T Q(0) X(t)$$

The result follows, using Proposition 3. ■

Appendix B. INTERMEDIATE LEMMAS USED IN SECTION 5.2

Lemma 8. Consider $\tilde{\Phi}$ the transition matrix associated with (2), which is denoted $\tilde{\Phi}(\cdot, D_1) = \tilde{\Phi}(\cdot)$ in the following for the sake of conciseness. The spatial-derivative of the distributed input (22) are, for $x \in [0, D_1/\hat{D}]$ and $x \in [D_1/\hat{D}, 1]$, respectively

$$\partial_x \hat{u}(x,t) = \partial_x \hat{w}(x,t) + K_1 \hat{D}/D_1 \times \partial_x \zeta\left(\hat{D}/D_1 x, t\right) + K_0 \hat{D} \Big[(\tilde{\Phi}(\hat{D}x)(A_0+BK_0) + \tilde{\Phi}(\hat{D}x - D_1)(A_1+BK_1))X(t) + D_1 \int_0^1 (\tilde{\Phi}(\hat{D}x - D_1 y)(A_0+BK_0) + \tilde{\Phi}(\hat{D}x - D_1(1+y)) \times (A_1+BK_1))A_1 \zeta(y,t) dy + \hat{D} \int_0^x (\tilde{\Phi}(\hat{D}(x-y))(A_0+BK_0) + \tilde{\Phi}(\hat{D}(x-y) - D_1)(A_1+BK_1))B\hat{w}(y,t)dy \Big]$$

$$\partial_x \hat{u}(x,t) = \partial_x \hat{w}(x,t) + K_1 \Bigg[(\tilde{\Phi}((\hat{D}-D_1)x)(A_0+BK_0) + \tilde{\Phi}((\hat{D}-D_1)x - D_1) \times (A_1+BK_1))X(t) + D_1 \int_0^1 (\tilde{\Phi}((\hat{D}-D_1)x - D_1 y) \times (A_0+BK_0) + \tilde{\Phi}((\hat{D}-D_1)x - D_1(1+y))(A_1+BK_1)) \times A_1 \zeta(y,t)dy + \hat{D}\int_0^x (\tilde{\Phi}((\hat{D}-D_1)x - \hat{D}y))(A_0+BK_0) \times (A_1+BK_1))B + \tilde{\Phi}((\hat{D}-D_1)x - \hat{D}y - D_1)\hat{w}(y,t)dy \Bigg] + K_0\hat{D}\Bigg[(\tilde{\Phi}(\hat{D}x)(A_0+BK_0) + \tilde{\Phi}(\hat{D}x - D_1)(A_1+BK_1))X(t) + D_1\int_0^1 (\tilde{\Phi}(\hat{D}x - D_1 y)(A_0+BK_0) + \tilde{\Phi}(\hat{D}x - D_1(1+y)) \times (A_1+BK_1))A_1\zeta(y,t)dy + \hat{D}\int_0^x (\tilde{\Phi}(\hat{D}(x-y))(A_0+BK_0) + \tilde{\Phi}(\hat{D}(x-y) - D_1)(A_1+BK_1))B\hat{w}(y,t)dy \Bigg]$$

Proof: The backstepping transformation (28) has the following inverse, for $x \in [0, D_1/\hat{D}]$

$$\hat{u}(x,t) = \hat{w}(x,t) + K_1\zeta\left(\frac{\hat{D}}{D_1}x,t\right) + K_0\Bigg[\tilde{\Phi}(\hat{D}x)X(t) + D_1\int_0^1 \tilde{\Phi}(\hat{D}x - D_1 y)A_1\zeta(y,t)dy + \hat{D}\int_0^x \tilde{\Phi}(\hat{D}(x-y))B\hat{w}(y,t)dy\Bigg]$$

and, for $x \in [D_1/\hat{D}, 1]$,

$$\hat{u}(x,t) = \hat{w}(x,t) + K_1\Bigg[\tilde{\Phi}((\hat{D}-D_1)x)X(t) + D_1\int_0^1 \tilde{\Phi}((\hat{D}-D_1)x - D_1 y)A_1\zeta(y,t)dy + \hat{D}\int_0^x \tilde{\Phi}((\hat{D}-D_1)x - D_1 y)B\hat{w}(y,t)dy\Bigg] + K_0\Bigg[\tilde{\Phi}(\hat{D}x)X(t) + D_1\int_0^1 \tilde{\Phi}(\hat{D}x - D_1 y)A_1\zeta(y,t)dy + \hat{D}\int_0^x \tilde{\Phi}(\hat{D}(x-y))B\hat{w}(y,t)dy\Bigg]$$

Taking a spatial-derivative of these two expressions, one obtains the Lemma statement. ■

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