CASING-HEADING PHENOMENON IN GAS-LIFTED WELL AS A LIMIT CYCLE OF A 2D MODEL WITH SWITCHES

Laure Sinègre * Nicolas Petit * Pierre Lemétayer ** Philippe Gervaud ** Philippe Ménégatti **

* *CAS, École des Mines de Paris, France*
** *CSTJF, TOTAL Exploration-Production, Pau, France*

Abstract: Oil well instabilities cause production losses. One of these instabilities, referred to as the "casing-heading" is an oscillatory phenomenon occurring on gas-lift artificially lifted well. This behavior is well represented by a 2D model with a vector field that is not continuously differentiable across several switching curves. These correspond to switches in flow rate functions describing the valves. In order to interpret the observed oscillations as a limit cycle we use the Poincaré-Bendixon theorem with a detailed study of uniqueness of trajectories and the derivation of a positive invariant set. Apart from the general case considered here, an illustrative example is given. The vector field is explicited and a similar limit cycle appears.

Keywords: Process Control, Dynamic Systems, Limit Cycles, Switching System, Gas-Lifted Well.

1. INTRODUCTION

Producing oil from deep reservoirs and lifting it through wells to the surface facilities often requires activation to maintain the oil output at a commercial level. In the gas-lift activation technique (Brown, 1973), gas is injected at the bottom of the well through the injection valve (point C in Figure 1) to lighten up the fluid column and to lower the gravity pressure losses. High pressure gas is injected at the well head through the gas valve (point A in Figure 1), then goes down into the annular space between the drilling pipe (casing, point B) and the production pipe (tubing, point C) where it enters. The oil produced from the reservoir (point F) and the injected gas mix in the tubing. They flow through the production valve E located at the surface.

Since 1986, a system for automatic handling of such wells, FCW (Full Control of Wells) has been developed by TOTAL. Wells have been operated by FCW since 1988. This tool schedules the opening of valves A and E following a sequential logic algorithm which steers the system to a prescribed setpoint. These can be stable or unstable. Details can be found in (Lemetayer and Miret, 1991).

High yield setpoints (low gas and high oil output) lie in an unstable region (Jansen *et al.*, 1999). A periodic phenomenon called "casing-heading" can appear. It consists of a succession of pressure build-up phases in the casing without production and high flow rate phases. These oscillations reduce the overall oil production and may damage the reservoir well interface and the facilities. Currently FCW does not fully address such dynamical instabilities.

This "casing-heading" instability is accurately represented by multiphase partial differential equations models (such as those implemented in Indiss™-IProd or Olga®2000). Yet, simpler mod-

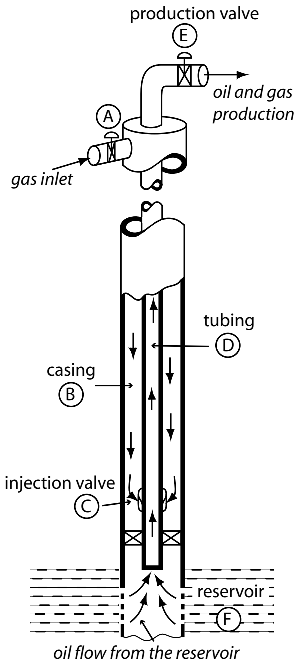

Fig. 1. Scheme of a gas-lift activated well.

The article is organized as follows. The system under consideration is presented in Section 2. In Section 3 a positive invariant set is constructed. In Section 4 existence and uniqueness of the trajectories are addressed through detailed studies around switching curves and their intersections. A future goal is to stabilize the system to the inner setpoint or to shrink the limit cycle. For that purpose a normalized sample problem is given for further reference. Its dynamics are explicited in Section 5. It exhibits a similar limit cycle. We hope it can serve as a test bench for various control techniques.

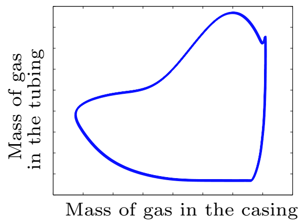

Fig. 2. Projection of a limit cycle obtained with the Indiss^{TM}-IProd multiphase simulator.

els can be used. In (Imsland, 2002; Eikrem *et al.*, 2003) a three balance ordinary differential equations model is used as the well dynamics. Numerical simulations prove the relevance of this approach. Further studies reveal that, as it is assumed that the gas mass fraction is constant with respect to the depth, the 3D model can be reduced to a 2D one (the masses of oil and gas in the tubing are highly correlated). This assumption eliminates possible instabilities due to propagation and thus let us focus on the casing-heading phenomenon. This representation is handy to interpret the casing-heading oscillations as a limit cycle. The contribution of this paper is to explain the observed planar limit cycle (e.g see Figure 2 for a sample Indiss^{TM}-IProd multiphase well simulation − exact scales are omitted for confidentiality reasons) through the Poincaré-Bendixon theorem. This system is related to other work on hybrid systems, such as the two-tank example addressed in (Hiskens, 2001), or the generalization of the Poincaré-Bendixon theorem to planar hybrid systems by (Simić *et al.*, 2002). Yet, several specific issues have to addressed here. The model includes two switching curves. These model the flow rate through the two valves (A and E). According to classic Saint-Venant laws (refer to (*Standard Handbook of Petroleum and Natural Gas Engineering*, 1996)) the flow rate is non differentially smooth around zero. The model is thus non differentially smooth across the switching curves. Therefore proving existence and uniqueness of the trajectories requires special care and does not directly derive from a Lipschtiz-continuity assumption.

2. DYNAMICS DEFINITION

2.1 Notations

We represent the behavior of the well around an unstable setpoint by the following dynamics over $[\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] \subset \mathbb{R}^+ \times \mathbb{R}^+$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \varepsilon w_{gc}(x) - w_{iv}(x,y) \\ w_{iv}(x,y) - \mu w_{pg}(y) \end{pmatrix} \quad (1)$$

We note $\mathcal{X} \triangleq [\underline{x}, \overline{x}]$, $\mathcal{Y} \triangleq [\underline{y}, \overline{y}]$, $X \triangleq (x,y)^t$ and $\dot{X} = F(X) = (F_1(x), F_2(x))^T$. This 2D dynamics is a restriction of the 3D one defined in (Eikrem *et al.*, 2003). w_{gc}, w_{iv} and w_{pg} are the gas flow rate through the gas valve A, through the injection valve C and through the production valve E. x and y represent the mass of gas in the casing and in the tubing. The positive parameters ε and μ stand for the openings of valves A and E. $\phi(\cdot, X_0)$ denote the solution of Equation (1) with X_0 as initial condition.

2.2 Hypothesis

We assume that both w_{iv} and w_{pg} vanish over their definition intervals. Let $\partial\mathcal{F}^o_{iv}$ and $\partial\mathcal{F}^o_{pg}$ be the boundaries of the sets $w_{iv}^{-1}(0)$ and $w_{pg}^{-1}(0)$. We assume the following hypothesis hold.

(H1) $w_{gc} : \mathbb{R} \to \mathbb{R}$ is C^1, strictly decreasing and does not vanish.

(H2) $w_{iv} = g_{iv} \circ \tau_{iv}$

- $\tau_{iv} : \mathbb{R}^2 \to \mathbb{R}$ is C^2, and strictly increasing w.r.t x and y.

- $g_{iv} : \mathbb{R} \to \mathbb{R}^+$, is C^0, strictly increasing over $\mathbb{R}^+$, C^1 over $\mathbb{R}/\{0\}$, and non Lipschitz at 0. $g_{iv}(0) = 0$. g'_{iv} is decreasing over $\mathbb{R}^+\backslash\{0\}$. $g'_{iv} \sim t^\lambda$ with $-1/2 < \lambda < 0$.

(H3) $w_{pg} = g_{pg} \circ \tau_{pg}$
- $\tau_{pg} : \mathbb{R}^2 \to \mathbb{R}$ is C^1, strictly increasing w.r.t. y, and does not depend on x.
- $g_{pg} : \mathbb{R} \to \mathbb{R}^+$, is C^0, strictly increasing over $\mathbb{R}^+$ and C^1 over $\mathbb{R}/\{0\}$, non Lipschitz at 0. $g_{pg}(0) = 0$.

(H4) τ_{iv} and τ_{pg} vanish over $\mathcal{X} \times \mathcal{Y}$. We define $\partial\mathcal{F}^o_{iv} \triangleq \tau^{-1}_{iv}(0)$ and $\partial\mathcal{F}^o_{pg} \triangleq \tau^{-1}_{pg}(0)$.

In order to construct a polygon $\mathcal{P}$ such as defined later on in Section 3.1 we need some further assumptions.

(H5) $\forall x \in \mathcal{X}$, $\dot{y}(x, \overline{y}) < 0$
(H6) $\dot{x}(\overline{x}, y_{pg}) < 0$
(H7) $\forall x \in \mathcal{X}$, $\tau_{iv}(x, \underline{y}) \leq 0$
(H8) $\forall y \in \mathcal{Y}$, $\tau_{iv}(\underline{x}, \underline{y}) \leq 0$

where, thanks to the continuity of w_{pg}, $y_{pg} \triangleq \max\{y/w_{pg}(y) = 0\}$.

One last assumption (H9) is that a constant K uniquely defined later on in Section 4.3 by the functions above is not zero.

2.3 Existence conditions of a limit cycle

Let $\Omega(\phi)$ be the limit set of ϕ. According to the Poincaré-Bendixon theorem as expressed in (Miller and Michel, 1982), the fact that $\Omega(\phi)$ contains no critical point combined to the uniqueness of the solution of Equation (1) is sufficient to guarantee the existence of a limit cycle. On the other hand, exhibiting a positive invariant set containing no stable equilibrium implies that $\Omega(\phi)$ contains no critical point. Therefore we can simply check that

- there exists a positive invariant set (this will be shown in Section 3),
- given an particular initial condition the solution is uniquely defined (this will be addressed in Section 4).

3. POSITIVE INVARIANCE

3.1 Some useful lemmas

Let $\mathcal{P}$ be a polygon ($(P_i)_{i\in[1,N]}$ its vertexes) such that

$$\forall i \in [1, N], \exists\lambda \text{ such that } \overrightarrow{P_iP_{i+1}} = \lambda F(P_i) \quad (2)$$

Classically, $\mathcal{P}$ is a positive invariant set if and only if

$$\forall X_0 \in \partial\mathcal{P}, \exists t > 0 \text{ s.t. } \forall\epsilon \in [0, t] : \phi(\epsilon, X_0) \in \mathcal{P} \quad (3)$$

Lemma 1. Assume that F is C^n on a neighborhood of X_0, with $X_0 \in [P_i, P_{i+1}]$. Define $u = \frac{P_1 \times P_2}{\|P_1\times P_2\|}$. If there exists $k \in [1, n]$ s.t.

$$\begin{cases} F(P_i)\times\dfrac{d^j\phi}{dt^j}(0, X_0) \cdot u = 0, j = 1..k-1 \\ F(P_i)\times\dfrac{d^k\phi}{dt^k}(0, X_0) \cdot u > 0 \end{cases}$$

then condition (3) holds.

Proof 1. A sufficient condition for condition (3) to be satisfied is that

$$\overrightarrow{P_iP_{i+1}} \times \overrightarrow{P_i\phi(\epsilon, X_0)} \cdot u > 0$$

This is equivalent to

$$A(\epsilon, X_0) = F(P_i) \times \overrightarrow{X_0\phi(\epsilon, X_0)} \cdot u > 0 \quad (4)$$

Since F is C^n on a neighborhood of X_0, an expansion of $A(\cdot, X_0)$ is

$$A(\epsilon, X_0) = \epsilon^{k-1}(F(P_i) \times \frac{d^k\phi}{dt^k}(0, X_0) \cdot u + o(1))$$

Therefore $A(\cdot, X_0)$ is strictly positive and condition (3) is satisfied. ∎

Similarly one can prove that

Lemma 2. Let $X_0 \in [P_i, P_{i+1}]$ and $(j, l) \in \{(1, 2); (2, 1)\}$. Assume that $F_j(P_i) = 0$. If F_l is continuous around X_0 and F_j is C^1, a sufficient condition leading to (3) is

$$\left.\begin{array}{r} (-1)^j\dot{x}_l(P_i)\dot{x}_j(X_0) > 0 \text{ or} \\ \left\{\begin{array}{r} \dot{x}_l(P_i)\dot{x}_j(X_0) = 0 \\ (-1)^j\dot{x}_l(P_i)\ddot{x}_j(X_0) > 0 \end{array}\right. \end{array}\right\} \quad (5)$$

Corollary 1. If $F_j(P_i) = 0$ and if F_j and F_l are only C^0, a more restrictive condition is

$$(-1)^j\dot{x}_l(P_i)\dot{x}_j(X_0) > 0$$

3.2 Positive invariant set candidate

Two curves play a key role in the construction of the candidate rectangle $\mathcal{P} = (P_1P_2P_3P_4)$. These are the set $\{(x, y)/\ \dot{x} = 0\}$ and the set $\{(x, y)/\ \dot{y} = 0\}$. We show that this rectangle, which is illustrated in Figure 3, satisfies Equation (2).

P_1, P_2 and P_3 construction Let ψ be defined by

$$\psi(x) \triangleq \varepsilon w_{gc}(x) - w_{iv}(x, y_{pg})$$

From ($H6$) and ($H8$), $\psi(\underline{x}) > 0$ and $\psi(\overline{x}) < 0$. Since ψ is continuous, increasing, we can uniquely define

$$x_1 = \max\{x/\psi(x) = 0\}$$

We note $P_1 \triangleq (x_1, y_{pg})$. At that point $\dot{x}$ and w_{pg} vanish. Further, similar arguments relying on (H5), and (H2)-(H8) respectively, uniquely define $P_2 \triangleq (x_1, y_2)$ with $y_2 \triangleq \min\{y/\dot{y}(x_1, y) = 0\}$ and $P_3 \triangleq (x_3, y_2)$ with $x_3 \triangleq \max\{x/\dot{x}(x, y_2) = 0\}$.

P_4 construction Let $P_4 \triangleq (x_3, y_{pg})$. $[P_3, P_4]$ is tangent to the field at P_3. Further, $[P_4, P_1]$ is tangent to the field at P_4. This arises from the the following argument. As w_{iv} is cancelling at $(\underline{x}, y_{pg})$ and strictly positive at P_1, we can choose ε parameter in Equation (1) such that $[P_4, P_1] \cap \partial\mathcal{F}^o_{iv} \neq \varnothing$. Therefore $w_{iv}(P_4) = 0$. As a consequence $\dot{x}(P_4) > 0$ and $\dot{y}(P_4) = 0$.

3.3 Intersections with switching lines

Let $X^2_{iv} \triangleq (x_{iv}, y_{pg})$ with $x_{iv} = \max\{x/(x, y_{pg}) \in [P_4, P_1] \cap \partial\mathcal{F}^o_{iv}\}$. Remembering that $w_{iv}(P_3) = \varepsilon w_{gc}(P_3) > 0$ we conclude $[P_3, P_4] \cap \partial\mathcal{F}^o_{iv} \neq \varnothing$. We note $X^1_{iv} \triangleq (x^3, y_{iv})$ with $y_{iv} \triangleq \max\{y/(x^3, y) \in [P_3, P_4] \cap \partial\mathcal{F}^o_{iv}\}$.

3.4 Positive invariance

Let X_0 be a point on the side of the rectangle. We want to prove that the trajectory $\phi(\cdot, X_0) = (\phi_x, \phi_y)^t$ starting at X_0 remains inside $\mathcal{P}$ for $t > 0$. We assume that trajectories are uniquely defined, this will be proven at Section 4.

Using Lemma 2 at points where F_2 is not C^1 Let $X_0 \in [P_1, P_2]$. F_1 vanishes at P_1, so F_1 being C^1 and F_2 only continuous around X_0 will complete the list of hypothesis needed to apply Lemma 2. F_2 is continuous by definition and F_1 is C^1, because $\forall X_0 \in [P_1, P_2]$

$$w_{iv}(X_0) \leq w_{iv}(P_1) = \varepsilon w_{gc}(P_1) > 0$$

Therefore checking condition (5) of Lemma 2 will prove that the trajectory starting at X_0 goes inside ($\mathcal{P}$). If $X_0 \in]P_1, P_2]$ the condition rewrites $-\dot{y}(P_1)\dot{x}(X_0) > 0$. As $-w_{iv}$ is decreasing w.r.t. y, $\dot{x}(X_0) < 0$. Adding that $\dot{y}(P_1) > 0$ ensures that the condition holds. If $X_0 = P_1$ the condition rewrites $-\dot{y}(P_1)\ddot{x}(X_0) > 0$. As $\ddot{x}(X_0) = -\partial_y w_{iv}(X_0)\dot{y}(X_0) < 0$ this condition holds. Following along the same lines it is easy to check that Lemma 2 can be applied at every point of $\partial\mathcal{P}$ except X^1_{iv} and $[P_4, P_1]$. At these points the C^1 condition is not verified. Notice also that at each vertex two conditions have to be verified, one for each side.

Using Corollary 1 at points where F_1 and F_2 are only C^0 When X_0 is an element of $X^1_{iv} \cup]X^2_{iv}, P_1]$ none of F coordinates vanish, therefore we can simply use the fact that F is continuous to apply Corollary 1. So for $X_0 = X^1_{iv}$ the condition is $-\dot{x}_2(P_3)\dot{x}_1(X_0) > 0$ which is easily checked. At $X_0 \in]X^2_{iv}, P_1]$ the condition is $\dot{x}_1(P_4)\dot{x}_2(X_0) > 0$.

A proof by contradiction when $X_0 \in [P_4, X^2_{iv}]$ Neither Lemma 2 (F_2 is not C^1) nor Corollary 1 ($\dot{y}(X_0) = 0$) can be used here. Yet, we can prove that a solution starting at X_0 cannot go below $y = y^{pg}$. Assume that there exists t_2 such that $\phi_y(t_2) < x^{pg}_2$, define t_1 such that

$$\begin{cases} \forall t \in]t_1, t_2],\ \phi_y(t) < x^{pg}_2 \\ \phi_y(t_1) = x^{pg}_2 \end{cases} \quad (6)$$

Refering to the mean value theorem $\phi_y(t_2) = \phi_y(t_1) + (t_2 - t_1)\phi'_y(t_c)$ with $t_c \in [t_1, t_2]$. $\phi'_y(t_c) = 0$ implies $\phi_y(t_2) = \phi_y(t_1)$ which contradicts (6). Finally, as the trajectory starting at $X_0 \in \partial\mathcal{P}$ satisfies condition (3), $\mathcal{P}$ defines a positive invariant set.

4. EXISTENCE AND UNIQUENESS OF THE TRAJECTORIES

The first hypothesis required by the Poincaré-Bendixon theorem is the existence and forward uniqueness of the solutions. Existence of a solution of (1) starting at $X_0 \in \mathcal{X} \times \mathcal{Y}$ follows from the continuity of F. Uniqueness of a solution of (1) starting at $X_0 \in (\mathcal{X} \times \mathcal{Y})/(\partial\mathcal{F}^o_{iv} \cup \partial\mathcal{F}^o_{pg})$ follows from the differentiable continuity of F around X_0.

4.1 Decoupling

Consider $X_0 \in [P_4, X^2_{iv}[\subset \partial\mathcal{F}^o_{pg}$. w_{iv} is null at P_1 and increasing with respect to x, so it cancels over $[P_4, X^2_{iv}]$. In a neighborhood of any point of this segment the system is decoupled. At this point the system writes

$$\begin{cases} \dot{x}(X_0) = \varepsilon w_{gc}(x_0) \\ \dot{y}(X_0) = -\mu w_{pg}(y_0) \end{cases}$$

Both right hand sides are decreasing functions because w_{pg} is increasing and w_{gc} is decreasing. Thus the solution starting at X_0 is unique (see (Brauer and Nohel, 1989)).

Let X_0 be $\in \partial\mathcal{F}^o_{iv}$, such that $F(X_0) \cdot \nabla\tau_{iv}(X_0) < 0$. Let ϕ be a solution starting at X_0. F being continuous and bounded in a neighborhood of X_0, we can define $T > 0$ such that $\forall t < T$, $X_0\phi(t) \cdot \nabla\tau_{iv}(X_0) > 0$. Therefore the solutions of (1) are the solutions of the decoupled system

$$\begin{cases} \dot{x} = \varepsilon w_{gc}(x) \\ \dot{y} = -\mu w_{pg}(y) \end{cases}$$

Each equation has a unique solution, so there exists a unique solution starting at X_0.

4.2 Transversality argument

Let $X_0 \in \{X \in \partial\mathcal{F}^o_{iv}/F(X) \cdot \nabla\tau_{iv}(X) > 0\} \cup [X^2_{iv}, P_1]$. Rewriting dynamics (1) in the (y, z) coordinates, with $z = \tau_{iv}(x, y)$, yields

$$\begin{cases} \dot{z} = F(\xi(y,z),y) \cdot \nabla \tau_{iv}(\xi(y,z),y) \\ \dot{y} = g_{iv}(z) - \mu w_{pg}(y) \end{cases} \quad (7)$$

where ξ is a C^2 function defined from the implicit function theorem applied to $z = \tau_{iv}(\xi(y,z),y)$. The decoupling argument does not hold anymore, but we can use the transversality property at 0, $\dot{z}$ is strictly positive, therefore $\exists \alpha^-, \alpha^+, T \in \mathbb{R}^+ \backslash \{0\}$ such that $\forall t \in [0,T]$

$$z_0 + \alpha^- t \leq z(t) \leq z_0 + \alpha^+ t \quad (8)$$

When $y_0 = \underline{y}$ and $z_0 \neq 0$, $\dot{y}(0)$ is strictly positive which allow us to define $\beta^-, \beta^+, T \in \mathbb{R}^+ \backslash \{0\}$

$$y_0 + \beta^- t \leq y(t) \leq y_0 + \beta^+ t \quad (9)$$

Now consider two distinct solutions (y_1, z_1) and (y_2, z_2), let $e_y \triangleq y_2 - y_1$ and $e_z \triangleq z_2 - z_1$. The key of the proof is to use equation (8) to define an upper-bound to $|e| = |(e_y, e_z)|$. From (8) and (9) we deduce that $\forall t \in]0,T]$ $y(t) > y_0$ and $z(t) > 0$. Therefore the solution of (7) starting at that point is unique. In the case of $(y_0, z_0) = (\underline{y}, 0)$ this property still holds. The two solutions $(\underline{y}_1, z_1)$ and (y_2, z_2) cannot split but at $t = 0$. Furthermore we define T' such that e_y, e_z and their derivatives remain positive over $]0,T']$. The dynamics rewrites as Equation (10). We replace the C^1 functions $\partial_x \tau_{iv}$, $\partial_y \tau_{iv}$ and w_{gc} by their first order expansion around X_0 in the first equation of (10)

$$\dot{z} = A - B g_{iv}(z) - C \mu w_{pg}(y) + Dz + Ey + R(y,z) \quad (11)$$

With $A > 0$, $C > 0$ and

$$\lim_{(y,z) \to (y_0,0)} \frac{R(y,z)}{|(y,z) - (y_0,0)|} = 0 \quad (12)$$

Using the mean value theorem, we can define $(y_c, y_c', y_c'') \in [y_1, y_2]$ and $(z_c, z_c', z_c'') \in [z_1, z_2]$ such that the dynamics of e is

$$\begin{cases} \dot{e}_y = - \mu w_{pg}'(y_c) e_y + g_{iv}'(z_c) e_z \\ \dot{e}_z = (-C\mu w_{pg}'(y_c') + E + \partial_y R(y_c'', z_2)) e_y \\ \qquad + (-B g_{iv}'(z_c') + D + \partial_z R(y_1, z_c'')) e_z \end{cases} \quad (13)$$

Recalling (12) one can define T', k and k' such that over $]0,T']$

$$\dot{e}_z \leq (-C\mu w_{pg}'(y_c') + kE) e_y + (-B g_{iv}'(z_c') + k'D) e_z$$

To define the upper-bound of (13), we recall the transversality argument. g_{iv}' being monotonous we deduce

$$\begin{cases} 0 \leq \dot{e}_y \leq g_{iv}'(z_0 + \alpha^\pm t) e_z \\ 0 \leq \dot{e}_z \leq kE e_y + (-B g_{iv}'(z_0 + \alpha^\pm t) + k'D) e_z \end{cases} \quad (14)$$

Notice that for $z_0 > 0$ we do not need the linear bounds of (8) to derive a proper upper-bound in (14). Yet, for $z_0 = 0$ the upper-bound goes to infinity, therefore we use that $\dot{z}(0)$ is not zero. Remark also that this kind of hypothesis is not required for $\dot{y}$. Integrating between s and t ($t < \min(t', t'')$ and $s > 0$) gives

$$e(t) \leq \int_s^t A(u) e(u) du + e(s)$$

$$\text{with } A(t) = \begin{pmatrix} 0 & g_{iv}'(z_0 + \alpha^\pm t) \\ kE & (-B g_{iv}'(z_0 + \alpha^\pm t) + k'D) \end{pmatrix}$$

Using $|A| = \sum_{i,j=1}^2 |a_{ij}|$ we deduce

$$|e(t)| \leq \int_s^t |A(u)||e(u)| du + |e(s)|$$

Therefore the Gronwall inequality theorem((Brauer and Nohel, 1989)) yields

$$|e(t)| \leq |e(s)| \exp\left(\int_s^t |A(u)| du \right) \quad (15)$$

As the exponential term is bounded, the limit of the right-hand side of equation (15) is also 0 when s goes to 0 which concludes the proof.

4.3 Non transverse case

Define X_0 such that $X_0 \in \partial \mathcal{F}_{iv}^o$ and $F(X_0) \cdot \nabla \tau_{iv}(X_0) = 0$. The initial conditions of equation (7) become $\dot{z}(0) = z(0) = 0$, $\dot{y}(0) < 0$ and $y(0) > y_{pg}$. In inequality (8), $\dot{z}(0) = 0$ yields $\alpha^\pm = 0$. The upper-bound $|A(u)|$ goes to infinity as u goes to zero. System (14) does not give further result. Yet, using $y \sim y_0 + \dot{y}(0)t$, Equation (11) yields

$$\dot{z} \sim Kt - B g_{iv}(z)$$

with

$$K = (E - C\mu w_{pg}'(y_0)) \quad (16)$$

The role of assumption (H9) appears here as a substitute to the transversality property of Section 4.2. It implies that when the field is tangent to the switching curve there exists a non vanishing higher order forcing term (which actually arises from the coupling of the y dynamics onto the z dynamics). Using L'Hospital's rule we find that Kt is the predominant term. Thus, for a given K, the solutions are positive or negative exclusively. Therefore, if $K < 0$ we use the decoupling argument to conclude to uniqueness. If $K > 0$ we use $z \sim Kt^2/2$ instead. As $t \mapsto g_{iv}'(t^2)$ is integrable around 0 the exponential term of the right-hand side of Equation (15) is bounded, therefore letting s go to zero yields $e(t) = 0$.

4.4 Conclusion

Away from $\partial \mathcal{F}_{iv}^o \cup \partial \mathcal{F}_{pg}^o$ uniqueness follows from the differentiable continuity of F. Points at which the field points toward the $\tau_{iv} < 0$ zone were studied in Section 4.1 where a decoupling argument

$$\begin{cases} \dot{z} = \partial_x \tau_{iv}(\xi(y,z),y)(\varepsilon w_{gc}(\xi(y,z)) - g_{iv}(z)) + \partial_y \tau_{iv}(\xi(y,z),y)(g_{iv}(z) - \mu w_{pg}(y)) \\ \dot{y} = g_{iv}(z) - \mu w_{pg}(y) \end{cases} \quad (10)$$

was used. Otherwise, when available, transversality was used (see Section 4.2). Finally, the case of a field tangential to $\partial\mathcal{F}^o_{iv}$ was addressed in Section 4.3. All cases being addressed, uniqueness is proven.

5. A CASE STUDY

While appearing as a limit case of our result (see (H2)), square roots are often used for valve modelling. Uniqueness proof follows along the exact same lines except for the final points addressed in Section 4.3. Instructively, an alternative study leads to the conclusion. Let $\mathcal{X} = \mathcal{Y} \triangleq [5/4 - \sqrt{13/8}, 5/4]$, $\varepsilon = 0.1$ and $\mu = 2$. Let

$$w_{gc}(x,y) \triangleq \sqrt{2-x}$$
$$\tau_{iv}(x,y) \triangleq 13/8 - (x-5/4)^2 - (y-5/4)^2$$
$$\tau_{pg}(x,y) \triangleq y^{\frac{3}{2}}$$

with $g_{iv} = g_{pg} \triangleq \sqrt{\max(0,\cdot)}$. Equilibrium points are unstable with positive real part complex conjugate poles. Hypothesis (H1), (H2), (H3), (H4) are verified. Let us check hypothesis (H5), (H6), (H7) and (H8) (with $y_{pg} = 0$)

$(H5)$ $\forall x \in \mathcal{X}$, $\dot{y}(x, \frac{5}{4}) = \sqrt{\frac{13}{8} - \left(x - \frac{5}{4}\right)^2} - 2\left(\frac{5}{4}\right)^{\frac{3}{4}} < 0$

$(H6)$ $\dot{x}(5/4, 0) = 0.1\sqrt{3}/2 - 1/4 < 0$

$(H7)$ $\forall x \in \mathcal{X}$, $\tau_{iv}\left(x, \frac{5}{4} - \sqrt{\frac{13}{8}}\right) = -(x - \frac{5}{4})^2 \leq 0$

$(H8)$ $\forall y \in \mathcal{Y}$, $\tau_{iv}\left(\frac{5}{4} - \sqrt{\frac{13}{8}}, y\right) = -(y - \frac{5}{4})^2 \leq 0$

These hypothesis are also verified. Yet, $\alpha = 1/2$, thus we substitute Section 4.3 with the following study. Around $X_0 = (y_0, 0)$ where the field is tangent to $\partial\mathcal{F}^o_{iv}$ we have, $y \sim y_0 + \dot{y}(0)t$ ($\dot{y}(0) < 0$). Equation (11) now yields

$$\dot{z} \sim -B\sqrt{z} + Kt$$

With $B = -1.93$ and $K = (E - C\mu 3/4 y_0^{-1/4})\dot{y}(0) = 0.503$. Using L'Hospital's rule we compute: $z(t) \sim at^2$, with $a = 1.38$. As $|e(s)| = \circ(s^2)$, Equation (12) becomes

$$|e(t)| \leq \circ(1) e^{b(t-s) + (2 - \frac{1-B}{2\sqrt{a}}) \ln \frac{t}{s}}$$

As $2 - (1-B)/(2\sqrt{a}) = 0.757$, letting s go to 0 implies that $e(t) = 0$. Uniqueness is proven. Figure 3 shows the construction of the positive invariant set and the limit cycle.

Acknowledgments We would like to thank D. E. Chang and L. Praly for their helpful insights regarding properties of this dynamical system.

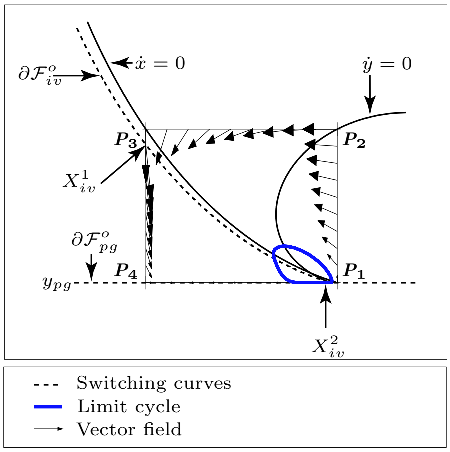

Fig. 3. Limit cycle and positive invariant set for the sample problem.

REFERENCES

Brauer, F. and J.A. Nohel (1989). *The qualitative theory of ordinary differential equations. An introduction.* Dover Publications, INC., New York.

Brown, K. E. (1973). *Gas Lift Theory and Practice.* Petroleum publishing CO., Tulsa, Oklahoma.

Eikrem, G.O., L. Imsland and B. Foss (2003). Stabilization of gas-lifted wells based on state estimation. In: *Preprints of the ADCHEM 2003, International Symposium on Advanced Control of Chemical Processes.*

Hiskens, I.A. (2001). Stability of limit cycles in hybrid systems. In: *Proc. of the 34th Hawaii International Conf. on System Sciences.*

Imsland, L.S. (2002). Topics in Nonlinear Control - Ouput Feedback Stabilization and Control of Positive Systems. PhD thesis. Norwegian University of Science and Technology, Department of Engineering and Cybernetics.

Jansen, B., M. Dalsmo K. Havre, L. Nøkleberg, V. Kritiansen and P. Lemetayer (1999). Automatic control of unstable gas-lifted wells. *SPE Annual technical Confererence and Exhibition.*

Lemetayer, P. and P. M. Miret (1991). Tool of the 90's to optimize gas-lift efficiency in the Gonelle field, Gabon. *SPE Annual technical Confererence and Exhibition.*

Miller, R.K. and A.N. Michel (1982). *Ordinary differential equations.* Academic Press.

Simić, S.N., K.H. Johansson, J. Lygeros and S. Sastry (2002). Hybrid limit cycles and hybrid Poincaré-Bendixon. In: *Proc. of the 15th IFAC World Congress.*

Standard Handbook of Petroleum and Natural Gas Engineering (1996). 6 ed. Gulf Professional Publishing.