Optimal Control of Systems Subject to Input-Dependent Hydraulic Delays

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***Abstract*—In this article, we study the optimal control of systems subject to input-varying hydraulic delays, i.e., systems where the delay on the input depends on the past values of the input through a specific integral relation. The calculus of variations of this problem reveals its nondifferentiable nature. Then, a smooth relaxation is proposed to derive an iterative optimization algorithm. A convergence proof is detailed. The practical interest of the algorithm is evidenced on a numerical example.**

***Index Terms*—Input-dependent delays, optimal control, variable time delays.**

I. Introduction

One approach to control time-delay systems is to schedule the input signals so that the system outputs approach desired setpoints in finite time. Numerous researchers have studied this topic [1]–[6], by focusing on controllability and trajectory parameterization, in particular. Furthermore, several works have considered optimization of the transients, i.e., optimal control strategies. With constant delays, the mathematical formulations have long been known (see [7]–[14]). A detailed survey can be found in [15] and [16]. These works cover cases of multiple input and state delays, with state constraints, in the framework of Pontryagin's maximum principle [17]. On the application side, model predictive control (MPC) routinely handles linear systems with fixed delays, and industrial implementations in commercial software are commonplace in the process industries [18]–[21].

Interestingly enough, it appears that only little attention has been given to dynamic optimization problems under varying delays. Since the seminal work of [22], most research efforts have focused on closed-form solutions to LQR problems for dynamics impacted by time-varying delays, see [23]. However, these approaches usually do not consider cases where the delay variability actually depends on the input or the state. Indeed, in most practical applications where delays are *a priori* known to be variable, this information is simply ignored and the delays are assumed to be fixed. A noteworthy exception is the early work of [24] on state-dependent delays. Unfortunately, it does not seem to have received the appropriate attention, and its results have not been implemented in any available software package. In this article, we consider a class of structured delays denoted as *hydraulic delays*. Such delays are defined by the following relation:

$$\int_{r_u(t)}^{t} \phi(u(\tau))\,\mathrm{d}\tau = 1 \tag{1}$$

where u is a subset of the system control variables (typically flow rates or valve openings), ϕ is a strictly positive, scalar-valued smooth function, and $r_u(t) \triangleq t - D_u(t)$ with $D_u(t)$ the delay. This type of delay is the exact solution of a plug-flow transport equation [25]; hence, its designation is hydraulic [26]. A quasi-steady approximation is sometimes considered in which the delay is modeled as inversely proportional to a function of the input (see [27, (16)]). Problems where such delay dependence in the control is important are numerous and of practical importance in chemical, process, and combustion engineering (among others) to model mixing, recycle streams, and spatially distributed cascades of reactions. Examples of such systems can be found in [28]–[41]. From a historical viewpoint, the appearance of the integral (1) with a nonlinear function ϕ can be traced down to the pioneer works on rockets by [42], [43], where the sensitiveness of the delay to the pressure is identified as a main source of combustion instability, as noted in several occasions in [44, ch. 12].

The article is organized as follows. Given a general objective function to minimize, we carry out the calculus of variations for an input-independent time-varying delay and an input-dependent hydraulic delay under weak regularity assumptions in Section II. Doing so, we stress the structural differences between these two cases. The conditions on Gâteaux-differentiability are given in Theorem 1. To relate the nondifferentiability to a physical interpretation, an illustrative example is presented, along with a schematic picturing the root cause phenomenon. Then, we propose a regularized approximation of this problem, prove its smoothness and derive its stationarity conditions. In Section III, we present an iterative optimization algorithm to solve a subclass of the problems tackled in Section II with systems exhibiting input-dependent input delays. We lay out a detailed proof of convergence in Theorem 2. Finally, in Section IV, we present and

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discuss numerical results, based on a simple benchmark problem from [32], illustrating the performances of the algorithm.

Notations: Let $L^2(E,F)$ be the set of functions of integrable square on E with values in F. With $T>0$, let $\ell \in L^2([0;T],\mathbb{R}^n)$, we note $\|\ell\|_1$ and $\|\ell\|_2$ the L^1-norm 1 and L^2-norm of the function ℓ, respectively. Using Cauchy–Schwarz inequality, $\|\ell\|_1 \le \sqrt{nT}\|\ell\|_2$. We note $C^1_{\mathrm{pw}}([0;T],\mathbb{R}^p)$ the class of piecewise C^1 functions from $[0;T]$ to $\mathbb{R}^p$, having a finite number of jumps in their values or derivatives on their interval of definition. Let $D^1([0;T],\mathbb{R}^p)$ be the class of functions from $[0;T]$ to $\mathbb{R}^p$ that are differentiable but whose derivative is not necessarily continuous. We note $u \in \mathbb{R}^p$ the control and $x \in \mathbb{R}^m$ the state of the system. Let $\mathcal{L} : [0;T] \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ and $f : [0;T] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^m$ be smooth functions. Let $g \in L^2(\mathbb{R},\mathbb{R}^n)$ and $\mathcal{I}$ be an interval of $\mathbb{R}$. We note $g_{\mathcal{I}}$ the restriction of g to $\mathcal{I}$. We note $\mathbb{1}$ the indicator function. Let h be a function of a real variable, we note the one-sided limit whose s-argument defines how the t-argument is approached

$$\lim_{\substack{\tau \to t \\ s}} h(\tau) = \begin{cases} \lim\limits_{\tau \to t^+} h(\tau) \text{ if } 0 \le s \\ \lim\limits_{\tau \to t^-} h(\tau) \text{ if } 0 > s. \end{cases}$$

We note the function $\mathrm{sign} : \mathbb{R} \to \{-1;0;1\}$ that maps strictly positive (resp. negative) arguments to 1 (resp. -1) and 0 onto itself. For notational ease, given a fixed delayed time law r, we consider $[t,x,u] = (t, x(t), x(r(t)), u(t), u(r(t)))$ and, similarly, $[t,x,u]_v = (t, x(t), x(r_v(t)), u(t), u(r_v(t)))$.

II. Calculus of Variations With Time-Varying and Input-Dependent Delays

A. Fixed Time-Varying Delays

As a preliminary, consider r a smooth, strictly increasing function such that for all t, $r(t) < t$. This function defines a delayed time law. Take $(u_0, x_0) \in L^2([r(0);0],\mathbb{R}^p) \times D^1([r(0);0],\mathbb{R}^p)$. Consider the following optimal control problem having r as *fixed*[1] time-varying delay:

$$\mathcal{P}_r : \min_u \int_0^T \mathcal{L}(t,x(t),u(t))\,\mathrm{d}t + \psi(x(T)) \triangleq J_r(u)$$
$$\text{s.t.} \quad \dot{x}(t) = f([t,x,u]),\ x_{[r(0);0]} = x_0,\ u_{[r(0);0]} = u_0.$$

We seek to establish necessary stationarity conditions characterizing optimal solutions of $\mathcal{P}_r$. Following, e.g., [45] these conditions are equivalent to the stationarity conditions of the augmented functional, where the constraints of the dynamics have been adjoined, which, using an integration by parts, is

$$\begin{aligned}\bar{J}_r(x,u,\lambda) = \int_0^T & \mathcal{L}(t,x(t),u(t)) + \dot{\lambda}(t)^T x(t) \\ & + \lambda(t)^T f([t,x,u])\,\mathrm{d}t \\ & - \lambda(T)^T x(T) + \lambda(0)^T x(0) + \psi(x(T)).\end{aligned} \quad (2)$$

[1]*fixed* means that the function only argument is t, it does not depend on the input, even implicitly.

To compute the Gâteaux derivatives of $\bar{J}_r$ ([46]), given any $\delta \in \mathbb{R}$, x, u, λ, let us first consider the cost variation associated with a variation of its first argument in a direction h

$$\begin{aligned}&\bar{J}_r(x+\delta h,u,\lambda) - \bar{J}_r(x,u,\lambda) = \\ &\quad \delta \int_0^T \frac{\partial \mathcal{L}}{\partial x}(t,x(t),u(t))h(t) + \dot{\lambda}(t)^T h(t) \\ &\quad + \lambda(t)^T \frac{\partial f}{\partial x}([t,x,u])h(t) \\ &\quad + \lambda(t)^T \frac{\partial f}{\partial x_r}([t,x,u])h(r(t))\,\mathrm{d}t \\ &\quad - \delta\lambda(T)^T h(T) + \delta \frac{\partial \psi}{\partial x}(x(T))h(T) + o(\delta).\end{aligned}$$

This immediately leads to the expression of the Gâteaux derivative w.r.t. the x-argument

$$\begin{aligned}D_h\bar{J}_r(x) = \int_0^T & \frac{\partial \mathcal{L}}{\partial x}(t,x(t),u(t))h(t) + \dot{\lambda}(t)^T h(t) \\ & + \lambda(t)^T \frac{\partial f}{\partial x}([t,x,u])h(t) \\ & + \lambda(t)^T \frac{\partial f}{\partial x_r}([t,x,u])h(r(t))\,\mathrm{d}t \\ & - \lambda(T)^T h(T) + \frac{\partial \psi}{\partial x}(x(T))h(T).\end{aligned}$$

This last expression is not handy for the coming derivation of stationarity conditions because the expression under the integral sign mixes the values of h at both time t and time $r(t)$. Since h is an admissible variation, for all $t \le 0,\ h(t) = 0$. This gives a first simplification. Then, using a change of variables, one finds

$$\begin{aligned}&\int_0^T \lambda(t)^T \frac{\partial f}{\partial x_r}([t,x,u])h(r(t))\,\mathrm{d}t \\ &= \int_0^{r(T)} \lambda(r^{-1}(t))^T \frac{\partial f}{\partial x_r}\left([r^{-1}(t),x,u]\right)(r^{-1})'(t)h(t)\,\mathrm{d}t.\end{aligned}$$

Finally, this leads to

$$\begin{aligned}D_h\bar{J}_r(x) = \int_0^T & \left(\frac{\partial \mathcal{L}}{\partial x}(t,x(t),u(t)) + \dot{\lambda}(t)^T\right. \\ & + \lambda(t)^T \frac{\partial f}{\partial x}([t,x,u]) \\ & + \mathbb{1}_{[0;r(T)]}(t)(r^{-1})'(t) \cdot \lambda(r^{-1}(t))^T \\ & \left.\cdot \frac{\partial f}{\partial x_r}\left([r^{-1}(t),x,u]\right)\right) h(t)\,\mathrm{d}t \\ & + \left(-\lambda(T)^T + \frac{\partial \psi}{\partial x}(x(T))\right) h(T).\end{aligned}$$

Similarly, we establish the Gâteaux derivative w.r.t. the input $D_h\bar{J}_r(u)$ and the adjoint variables $D_h\bar{J}_r(\lambda)$. Any stationary solution (x^*,u^*,λ^*) of $\bar{J}_r$ is characterized by the relations

$$\forall(h_1,h_2,h_3),\ D_{h_1}\bar{J}_r(x^*) = D_{h_2}\bar{J}_r(u^*) = D_{h_3}\bar{J}_r(\lambda^*) = 0.$$

Then, using Dubois–Reymond lemma (see [45]), we establish the following result.

Proposition 1: Any locally optimal solution of $\mathcal{P}_r$ verifies the following two-point boundary value problem (TPBVP)

$$\begin{aligned}\dot{x}(t) &= f([t,x,u]), \quad x(0)=x_0\\ \dot{\lambda}(t) &= -\frac{\partial \mathcal{L}}{\partial x}(t,x(t),u(t))^T - \frac{\partial f}{\partial x}([t,x,u])^T\lambda(t)\\ &\quad - \mathbb{1}_{[0;r(T)]}(t)(r^{-1})'(t)\frac{\partial f}{\partial x_r}\left([r^{-1}(t),x,u]\right)^T\lambda(r^{-1}(t))\\ \lambda(T) &= \frac{\partial \psi}{\partial x}(x(T))^T\\ 0 &= \frac{\partial \mathcal{L}}{\partial u}(t,x(t),u(t))^T + \frac{\partial f}{\partial u}([t,x,u])^T\lambda(t)\\ &\quad + \mathbb{1}_{[0;r(T)]}(t)(r^{-1})'(t)\frac{\partial f}{\partial u_r}\left([r^{-1}(t),x,u]\right)^T\lambda(r^{-1}(t)).\end{aligned}$$

B. Input-Dependent Delays: A Nonsmooth Problem

From now on, the delay depends on the input signal according to (1).

1) Gâteaux Differentiability: Take $(u_0,x_0)\in C^1_{\text{pw}}([r_0;0],\mathbb{R}^p)\times D^1([r_0;0],\mathbb{R}^p)$, $r_0<0$ with

$$\int_{r_0}^{0}\phi\left(u_0(\tau)\right)\,\mathrm{d}\tau = 1. \tag{3}$$

Consider the optimal control problem with input-dependent delays

$$\begin{aligned}\mathcal{P}_0:\min_u \int_0^T &\mathcal{L}(t,x(t),u(t))\,\mathrm{d}t + \psi(x(T)) \triangleq J_0(u)\\ \text{s.t.}\quad &\dot{x}(t) = f(t,x(t),x(r_u(t)),u(t),u(r_u(t)))\\ &x_{[r_0;0]}=x_0,\ u_{[r_0;0]}=u_0\end{aligned}$$

where r_u is implicitly defined by (1). Before addressing the derivation of the optimality conditions, we introduce the following handy result.

Proposition 2: For any $t\in[0;T]$, $(u,h)\in C^1_{\text{pw}}([0;T],\mathbb{R}^p)^2$ and $s\in\{-1;1\}$, we have

$$\lim_{\underset{s}{\delta\to 0}}\frac{r_{u+\delta h}(t)-r_u(t)}{\delta} = \frac{1}{\lim\limits_{\underset{s'}{\tau\to r_u(t)}}\phi(u(\tau))}\int_{r_u(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \tag{4}$$

where $s'=\text{sign}(s\cdot\int_{r_u(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau)$. In particular, if u is continuous at $r_u(t)$, the Gâteaux derivative of $r_u(t)$ w.r.t. the input at point u in the direction h is

$$D_h r_u(t) = \frac{1}{\phi(u(r_u(t)))}\int_{r_u(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau.$$

Similarly, for any $t\in[0;r_u(T)[$

$$\lim_{\underset{s}{\delta\to 0}}\frac{r^{-1}_{u+\delta h}(t)-r^{-1}_u(t)}{\delta} = -\frac{1}{\lim\limits_{\underset{s'}{\tau\to r_u^{-1}(t)}}\phi(u(\tau))}\int_{t}^{r_u^{-1}(t)}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \tag{5}$$

where

$$s' = \text{sign}\left(-s\cdot\int_{t}^{r_u^{-1}(t)}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\right) \tag{6}$$

and if u is continuous at $r_u^{-1}(t)$, the Gâteaux derivative is

$$D_h r_u^{-1}(t) = -\frac{1}{\phi(u(r_u^{-1}(t)))}\int_{t}^{r_u^{-1}(t)}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau.$$

Proof: From (1), we have

$$1=\int_{r_u(t)}^{t}\phi(u(\tau))\,\mathrm{d}\tau = \int_{r_{u+\delta h}(t)}^{t}\phi(u(\tau)+\delta h(\tau))\,\mathrm{d}\tau.$$

Then, from the smoothness of ϕ, one deduces that

$$\int_{r_u(t)}^{r_{u+\delta h}(t)}\phi(u(\tau))\,\mathrm{d}\tau = \delta\int_{r_{u+\delta h}(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau + o(\delta). \tag{7}$$

Since we know *a priori* that $r_0\le r_{u+\delta h}(t)\le T$, one notices that the integral in the right-hand side is expressed on a bounded domain over which its argument is bounded and $\int_{r_u(t)}^{r_{u+\delta h}(t)}\phi(u(\tau))\,\mathrm{d}\tau \underset{\delta\to 0}{\longrightarrow} 0$. Recalling that $\phi>0$, we obtain the continuity of $r_u(t)$ w.r.t. the input $r_{u+\delta h}(t)-r_u(t)\underset{\delta\to 0}{\longrightarrow} 0$. Using this result with (7), we have

$$\frac{1}{\delta}\int_{r_u(t)}^{r_{u+\delta h}(t)}\phi(u(\tau))\,\mathrm{d}\tau = \int_{r_u(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau + o(1). \tag{8}$$

If

$$\int_{r_u(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \neq 0 \tag{9}$$

then (8) guarantees that, in a neighborhood of $\delta=0$,

$$\text{sign}(r_{u+\delta h}(t)-r_u(t)) = \text{sign}\left(\delta\int_{r_u(t)}^{t}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau\right).$$

Using this, the desired results (4) are finally obtained by taking alternatively the limit of (8) when δ goes to zero from above or below. Otherwise, when (9) fails, we directly get

$$\frac{1}{\delta}\int_{r_u(t)}^{r_{u+\delta h}(t)}\phi(u(\tau))\,\mathrm{d}\tau = o(1),\quad \lim_{\delta\to 0}\frac{r_{u+\delta h}(t)-r_u(t)}{\delta}=0.$$

The results regarding the variation of $r_u^{-1}(t)$ are established symmetrically after noticing that the definition of the delay implies that, for all $t \in [0; \min(r_u(t), r_{u+\delta h}(t))]$

$$1 = \int_t^{r_u^{-1}(t)} \phi(u(\tau))\,\mathrm{d}\tau = \int_t^{r_{u+\delta h}^{-1}(t)} \phi(u(\tau) + \delta h(\tau))\,\mathrm{d}\tau.$$

■

We now pursue our analysis. Let us derive the stationary points of the augmented functional

$$\bar{J}_0(x, u, \lambda) = \int_0^T \mathcal{L}(x(t), u(t)) + \lambda^T(t) f([t, x, u]_u) + \dot{\lambda}^T(t)x(t)\,\mathrm{d}t + \lambda^T(0)x(0) - \lambda^T(T)x(T) + \psi(x(T)).$$

By construction, there exists a finite number N of distinct time instants $r_0 < t_i \leq r_u(T)_{i=1\ldots N}$ at which the control input u is not smooth. For δ small enough, $u + \delta h$ has the same jumping points as u, plus those generated by δh, which will all have negligible contributions. The calculus of the Gâteaux derivative of $\bar{J}_0$ w.r.t. its second argument is decomposed over a mesh allowing us to cover both cases when the image of the jumps of u by the inverse perturbed delayed time law, $r_{u+\delta h}^{-1}(t)$, are each approached from below or above. This yields (10) shown at the bottom of this page, with

$$\Delta(t, \delta) = \lambda^T(t) \cdot \left(\frac{\partial f}{\partial x_r}([t, x, u]_u)\,\dot{x}(r_u(t)) \cdot \frac{r_{u+\delta h}(t) - r_u(t)}{\delta} + \frac{\partial f}{\partial u_r}([t, x, u]_u)\,\dot{u}(r_u(t)) \cdot \frac{r_{u+\delta h}(t) - r_u(t)}{\delta} + \frac{\partial f}{\partial u_r}([t, x, u]_u) \cdot h(r_{u+\delta h}(t)) \right)$$

where $\frac{\partial f}{\partial x_r}$, $\frac{\partial f}{\partial u_r}$ designate the partial derivatives of f w.r.t. its third and fifth arguments, respectively. Using (5) from Proposition 2, we know that on a neighborhood of $\delta = 0$, if the upper and lower Gâteaux derivatives of $r_u^{-1}(t_i)$ are nonzero at u, we have, with $\epsilon \triangleq \mathrm{sign}(r_{u+\delta h}^{-1}(t_i) - r_u^{-1}(t_i)) = \mathrm{sign}(-\delta \int_{t_i}^{r_u^{-1}(t_i)} \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau)$. The strict monotonicity of r_u and $r_{u+\delta h}$ gives that

$$r_u^{-1}(t_i) \leq t \leq r_{u+\delta h}^{-1}(t_i) \Rightarrow r_{u+\delta h}(t) \leq t_i \leq r_u(t). \tag{11}$$

and

$$r_{u+\delta h}^{-1}(t_i) \leq t \leq r_u^{-1}(t) \Rightarrow r_{u+\delta h}(t) \geq t_i \geq r_u(t). \tag{12}$$

Both of these inequalities (11) and (12) are instrumental for the evaluation of the integrals $\int_{\min(.)}^{\max(.)}(.)$ in (10), by determining the arguments of f as δ goes to zero. This gives

$$\begin{aligned} & o(1) + \frac{1}{\delta} \int_{\min(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))}^{\max(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))} \lambda^T(t) \cdot \left(f([t, x, u + \delta h]_{u+\delta h}) - f([t, x, u]_u) \right) \mathrm{d}t \\ &= \frac{1}{\delta} \int_{\min(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))}^{\max(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))} \lambda(t)^T \cdot \left(f\left(t, x(t), x(r_u(t)), \lim_{\tau \underset{\epsilon}{\to} r_u^{-1}(t_i)} u(\tau), \lim_{\tau \underset{-\epsilon}{\to} t_i} u(\tau)\right) \right. \\ &\quad \left. - f\left(t, x(t), x(r_u(t)), \lim_{\tau \underset{\epsilon}{\to} r_u^{-1}(t_i)} u(\tau), \lim_{\tau \underset{\epsilon}{\to} t_i} u(\tau)\right) \right) \mathrm{d}t. \end{aligned}$$

Otherwise, if the Gâteaux derivative of $r_u^{-1}(t_i)$ is equal to zero, we have

$$\lim_{\delta \to 0} \frac{1}{\delta} \int_{\min(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))}^{\max(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))} \lambda^T(t) \cdot \left(f([t, x, u + \delta h]_{u+\delta h}) - f([t, x, u]_u) \right) \mathrm{d}t = 0.$$

Finally, for $s \in \{-1; 1\}$, gathering the smooth and jump parts of the calculus and using (4) along with (5), we obtain

$$\begin{aligned} & \lim_{\delta \underset{s}{\to} 0} \frac{\bar{J}_0(u + \delta h) - \bar{J}_0(u)}{\delta} = \\ & \int_0^T \left(\frac{\partial \mathcal{L}}{\partial u}(t, x(t), u(t))h(t) + \lambda^T(t) \frac{\partial f}{\partial u}([t, x, u]_u)\,h(t) \right. \\ & + \lambda^T(t) \frac{\partial f}{\partial x_r}([t, x, u]_u) \cdot \frac{\dot{x}(r_u(t))}{\phi(u(r_u(t)))} \int_{r_u(t)}^t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \\ & + \lambda^T(t) \frac{\partial f}{\partial u_r}([t, x, u]_u) \cdot \frac{\dot{u}(r_u(t))}{\phi(u(r_u(t)))} \int_{r_u(t)}^t \frac{\partial \phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \\ & \left. + \lambda^T(t) \frac{\partial f}{\partial u_r}([t, x, u]_u)\,h(r_u(t)) \right) \mathrm{d}t \end{aligned}$$

$$\begin{aligned} \frac{\bar{J}_0(u + \delta h) - \bar{J}_0(u)}{\delta} = & \int_0^T \frac{\partial L}{\partial u}(t, x(t), u(t))h(t) + \lambda(t)^T \frac{\partial f}{\partial u}([t, x, u]_u) \cdot h(t)\,\mathrm{d}t + \int_0^{\min(r_u^{-1}(t_1), r_{u+\delta h}^{-1}(t_1))} \Delta(t, \delta)\,\mathrm{d}t \\ & + \sum_{i=1}^N \frac{1}{\delta} \int_{\min(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))}^{\max(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))} \lambda^T(t) \left(f([t, x, u + \delta h]_{u+\delta h}) - f([t, x, u]_u) \right) \mathrm{d}t \\ & + \sum_{i=1}^{N-1} \int_{\max(r_u^{-1}(t_i), r_{u+\delta h}^{-1}(t_i))}^{\min(r_u^{-1}(t_{i+1}), r_{u+\delta h}^{-1}(t_{i+1}))} \Delta(t, \delta)\,\mathrm{d}t + \int_{\max(r_u^{-1}(t_N), r_{u+\delta h}^{-1}(\min(t_N, r_{u+\delta h}(T))))}^T \Delta(t, \delta)\,\mathrm{d}t + o(1). \end{aligned} \tag{10}$$

$$+\sum_{i=1}^{N}\lambda(r_u^{-1}(t_i))^T \cdot\left(f\left(r_u^{-1}(t_i), x(r_u^{-1}(t_i)), x(t_i), \lim_{\substack{\tau\to r_u^{-1}(t_i)\\ s_i'}} u(\tau), \lim_{\substack{\tau\to t_i\\ -s_i'}} u(\tau)\right) - f\left(r_u^{-1}(t_i), x(r_u^{-1}(t_i)), x(t_i), \lim_{\substack{\tau\to r_u^{-1}(t_i)\\ s_i'}} u(\tau), \lim_{\substack{\tau\to t_i\\ s_i'}} u(\tau)\right)\right) \cdot \frac{s_i'}{\lim_{\substack{\tau\to r_u^{-1}(t)\\ s_i'}}\phi(u(\tau))}\int_{t_i}^{r_u^{-1}(t_i)}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \quad (13)$$

where $s_i' = s'(t_i)$ is given by (6).

Using (13), we can formulate the following result using $s_i'' = s''(t_i)$ where $s''(t) = \mathrm{sign}(-\int_t^{r_u^{-1}(t)} \frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau)$.

Theorem 1: Given (x,u,λ), $\bar{J}_0$ is Gâteaux differentiable w.r.t. its second argument at point (x,u,λ) in direction h iff (14) shown at the bottom of this page, holds.

Remark 1: Note that (14) does not trivially hold. When f is explicitly depending upon its delayed input, the augmented cost associated cannot be guaranteed to be differentiable w.r.t. the input at any given point if the function u is not continuous for $t>0$ (interestingly, however, discontinuities in the control prior to $t=0$ do not raise issues) and counter examples are straightforward to build (see Remark 2).

Remark 2: Consider (u,h) such that

$$u(t) = \begin{cases} 1 \text{ if } t\in[-1;0] \\ 2 \text{ if } t\in]0;0.5] \\ 3 \text{ if } t\in]0.5;1]\end{cases}$$

and

$$\forall t\in[0;1],\ h(t)=1$$

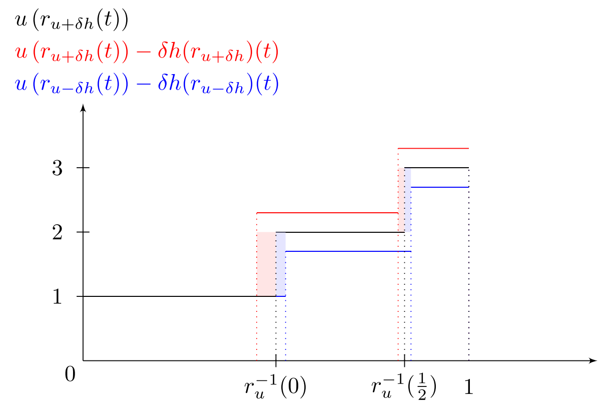


Fig. 1. First-order variation of the integral cost with a variation of delay due to a perturbation of u shows a dissymmetry due to the nondifferentiability of $r_u^{-1}(0)$.

along with the following functional $J_0(u)=\int_0^1 u(r_u(t))\,\mathrm{d}t$, where ϕ is the identity, $\phi(u)=u$, $\int_{r_u(t)}^t u(\tau)\,\mathrm{d}\tau = 1$. By choosing the discontinuities of u at $t=0$ and $t=0.5$ such that $r_u^{-1}(0)=0.5$, we create a nondifferentiability of r_u^{-1} with respect to u. This phenomena is illustrated in Fig. 1. More formally,

$$J_0(u) = \int_0^{r_u^{-1}(0)} 1\,\mathrm{d}t + \int_{r_u^{-1}(0)}^{r_u^{-1}(0.5)} 2\,\mathrm{d}t + \int_{r_u^{-1}(0.5)}^{1} 3\,\mathrm{d}t = \frac{5}{3}$$

where $r_u^{-1}(0)=0.5,\ r_u^{-1}(0.5)=\frac{5}{6}$. If $\delta<0$, then $r_{u+\delta h}^{-1}(t) > r_u^{-1}(t)$ and

$$\int_0^1 u(r_{u+\delta h}(t))\,\mathrm{d}t = \int_0^{r_u^{-1}(0)} 1\,\mathrm{d}t + \int_{r_u^{-1}(0)}^{r_{u+\delta h}^{-1}(0)} 1\,\mathrm{d}t + \int_{r_{u+\delta h}^{-1}(0)}^{r_u^{-1}(0.5)} 2\,\mathrm{d}t + \int_{r_u^{-1}(0.5)}^{r_{u+\delta h}^{-1}(0.5)} 2\,\mathrm{d}t + \int_{r_{u+\delta h}^{-1}(0.5)}^{1} 3\,\mathrm{d}t$$

$$\sum_{i=1}^{N}\lambda(r_u^{-1}(t_i))^T\left[f\left(r_u^{-1}(t_i), x(r_u^{-1}(t_i)), x(t_i), \lim_{\substack{\tau\to r_u^{-1}(t_i)\\ s_i''}} u(\tau), \lim_{\substack{\tau\to t_i\\ -s_i''}} u(\tau)\right) - f\left(r_u^{-1}(t_i), x(r_u^{-1}(t_i)), x(t_i), \lim_{\substack{\tau\to r_u^{-1}(t_i)\\ s_i''}} u(\tau), \lim_{\substack{\tau\to t_i\\ s_i''}} u(\tau)\right)\right]\frac{s_i''}{\lim_{\substack{\tau\to r_u^{-1}(t_i)\\ s_i''}}\phi(u(\tau))}\int_{t_i}^{r_u^{-1}(t_i)}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau$$
$$=\sum_{i=1}^{N}\lambda(r_u^{-1}(t_i))^T\left[f\left(r_u^{-1}(t_i), x(r_u^{-1}(t_i)), x(t_i), \lim_{\substack{\tau\to r_u^{-1}(t_i)\\ -s_i''}} u(\tau), \lim_{\substack{\tau\to t_i\\ s_i''}} u(\tau)\right) - f\left(r_u^{-1}(t_i), x(r_u^{-1}(t_i)), x(t_i), \lim_{\substack{\tau\to r_u^{-1}(t_i)\\ -s_i''}} u(\tau), \lim_{\substack{\tau\to t_i\\ -s_i''}} u(\tau)\right)\right]\frac{-s_i''}{\lim_{\substack{\tau\to r_u^{-1}(t_i)\\ -s_i''}}\phi(u(\tau))}\int_{t_i}^{r_u^{-1}(t_i)}\frac{\partial\phi}{\partial u}(u(\tau))h(\tau)\,\mathrm{d}\tau \quad (14)$$

and

$$J_0(u+\delta h) - J_0(u) = \int_{r_u^{-1}(0)}^{r_{u+\delta h}^{-1}(0)} (1-2)\ \mathrm{d}t + \int_{r_u^{-1}(0.5)}^{r_{u+\delta h}^{-1}(0.5)} (2-3)\ \mathrm{d}t + \delta \int_0^1 h(r_u(t))\,\mathrm{d}t + o(\delta).$$

Taking $s=-1$ in (5)

$$\lim_{\substack{\delta\to 0\\ s}} \frac{r_{u+\delta h}^{-1}(t) - r_u^{-1}(t)}{\delta} = -\frac{1}{\lim\limits_{\substack{\tau \to r_u^{-1}(t)\\ s'}} u(\tau)} \int_t^{r_u^{-1}(t)} 1\,\mathrm{d}\tau$$

with $s' = \mathrm{sign}(-s\int_t^{r_u^{-1}(t)} 1\,\mathrm{d}\tau) = 1$. Finally

$$\begin{aligned}\lim_{\delta\to 0^-} & \frac{J_0(u+\delta h) - J_0(u)}{\delta}\\ &= \frac{r_u^{-1}(0) - 0}{2} + \frac{r_u^{-1}(0.5) - 0.5}{3} + \frac{1}{2}\\ &= \frac{1}{4} + \frac{1}{9} + \frac{1}{2} = \frac{31}{36}.\end{aligned} \tag{15}$$

Conversely, if $\delta > 0$, then $r_{u+\delta h}^{-1}(t) < r_u^{-1}(t)$ and

$$J_0(u+\delta h) - J_0(u) = \int_{r_{u+\delta h}^{-1}(0)}^{r_u^{-1}(0)} (2-1)\ \mathrm{d}t + \int_{r_{u+\delta h}^{-1}(0.5)}^{r_u^{-1}(0.5)} (3-2)\ \mathrm{d}t + \frac{\delta}{2} + o(\delta)$$

and

$$\begin{aligned}\lim_{\delta\to 0^+} & \frac{J_0(u+\delta h) - J_0(u)}{\delta}\\ &= \frac{r_u^{-1}(0) - 0}{3} + \frac{r_u^{-1}(0.5) - 0.5}{3} + \frac{1}{2}\\ &= \frac{1}{6} + \frac{1}{9} + \frac{1}{2} = \frac{7}{9} < \frac{31}{36}.\end{aligned} \tag{16}$$

There is indeed a mismatch between the left and the right limits (15) and (16).

Consequently, $\mathcal{P}_0$ is actually a nonsmooth optimization problem and its optimal solutions cannot be characterized using the standard technique of imposing that all the variations of the augmented cost be equal to zero. This result also has important practical consequences. Indeed, any standard optimization technique requiring first (or second)-order regularity properties is expected to have difficulty solving $\mathcal{P}_0$. In particular, when trying to solve the problem using a discretization of the transport equation, it is expected that the Hessian should diverge as the spatial discretization is refined and the intrinsic nondifferentiability of the optimization problem is exposed.

C. Formulation of a Regularized Approximated Problem for Input-Dependent Delays

To overcome the mathematical difficulty stressed by Theorem 1, we consider a regularized version of $\mathcal{P}_0$ where the input u of the system having x as state is itself made to be the state of a pure integrator of an underlying input v. Take $(v_0, u_0) \in L^2([r_0;0],\mathbb{R}^p) \times D^1([r_0;0],\mathbb{R}^p)$, $r_0 < 0$ with (3) and

$$\forall t \in [r_0;0],\ u_0(t) = u_0(0) + \int_0^t v_0(\tau)\,\mathrm{d}\tau. \tag{17}$$

Let $P \in \mathcal{M}_p(\mathbb{R})$ be symmetric definite positive. The regularized optimal control problem is

$$\begin{aligned}\mathcal{P} : \min_v & \int_0^T \mathcal{L}(t,x(t),u(t)) + \frac{1}{2}v(t)^T P v(t)\,\mathrm{d}t \triangleq J(v)\\ \text{s.t.} \ & \dot{x}(t) = f(t,x(t),x(r_u(t)),u(t),u(r_u(t)))\\ & \dot{u}(t) = v(t)\\ & x_{[r_0;0]} = x_0,\ u_{[r_0;0]} = u_0,\ v_{[r_0;0]} = v_0.\end{aligned}$$

Carrying computations similar to those of Section II-B, we establish the following theorem.

Proposition 3: The stationarity conditions of $\mathcal{P}$ are given by the following TPBVP:

$$\begin{aligned}\dot{x}(t) &= f([t,x,u]_u), \quad x(0) = x_0\\ \dot{u}(t) &= v(t)\\ u_{[r_0;0]} &= u_0\\ \dot{\lambda}(t) &= -\frac{\partial \mathcal{L}}{\partial x}(t,x(t),u(t))^T - \frac{\partial f}{\partial x}([t,x,u]_u)^T \lambda(t)\\ &\quad - \mathbb{1}_{[t_0;r_u(t_0+T)]}(t)(r_u^{-1})'(t)\\ &\quad \cdot \frac{\partial f}{\partial x_r}([r_u^{-1}(t),x,u]_u)^T \cdot \lambda(r^{-1}(t))\\ \lambda(T) &= \frac{\partial \psi}{\partial x}(x(T))^T\\ \dot{\nu}(t) &= -\frac{\partial \mathcal{L}}{\partial u}(t,x(t),u(t))^T - \frac{\partial f}{\partial u}([t,x,u]_u)^T \lambda(t)\\ &\quad - \mathbb{1}_{[t_0;r(t_0+T)]}(t)(r_u^{-1})'(t)\\ &\quad \cdot \frac{\partial f}{\partial u_r}([r_u^{-1}(t),x,u]_u)^T \cdot \lambda(r^{-1}(t))\\ &\quad - \int_t^{r_u^{-1}(\min(t,r_u(T)))} \lambda(\tau)^T\\ &\quad \cdot \frac{\partial f}{\partial x_r}([\tau,x,u]_u)\frac{\dot{x}(r_u(\tau))}{\phi(u(r_u(\tau)))}\,\mathrm{d}\tau \cdot \frac{\partial \phi}{\partial u}(u(t))^T\\ &\quad - \int_t^{r_u^{-1}(\min(t,r_u(T)))} \lambda(\tau)^T\\ &\quad \cdot \frac{\partial f}{\partial u_r}([\tau,x,u]_u)\frac{v(r_u(\tau))}{\phi(u(r_u(\tau)))}\,\mathrm{d}\tau\,\frac{\partial \phi}{\partial u}(u(t))^T\\ \nu(T) &= 0, \quad 0 = Pv(t) + \nu(t).\end{aligned}$$

III. Numerical Resolution Algorithm

Let $x_0 \in \mathbb{R}^m$ and $P \in \mathcal{M}_p(\mathbb{R})$ be symmetric definite positive. Take $(v_0,u_0) \in L^2([r_0;0],\mathbb{R}^p) \times D^1([r_0;0],\mathbb{R}^p)$, $r_0 < 0$ with (3) and (17). Consider the following optimization problem, part of a subclass of the approximate regularized problem covered in Proposition 3 (no dependence on the past values of the state and, without loss of generality, no terminal cost)

$$\begin{aligned}\mathcal{P} : \min_{v} \int_0^T & \mathcal{L}(t, x(t), u(t)) + \frac{1}{2} v(t)^T P v(t)\, \mathrm{d}t \triangleq J(v) \\ \text{s.t.}\quad & \dot{x}(t) = f(t, x(t), u(t), u(r_u(t))) \\ & \dot{u}(t) = v(t),\ x(0) = x_0,\ u_{[r_0;0]} = u_0,\ v_{[r_0;0]} = v_0\end{aligned}$$

where r_u is implicitly defined by the relation and in particular $r_0 = r_u(0)$. Let us consider the operator $\mathfrak{P} : L^2([0;T], \mathbb{R}^p) \to D^1([0;T], \mathbb{R}^p) \times D^1([0;T], \mathbb{R}^m)^3$ such that $\mathfrak{P}(v) = (u, x, \lambda, \nu)$ is defined, according to Proposition 3, by

$$\dot{u}(t) = v(t),\ u_{[r_0;0]} = u_0 \tag{18}$$

$$\dot{x}(t) = f(t, x(t), u(t), u(r_u(t))),\ x(0) = x_0 \tag{19}$$

$$\begin{aligned}\dot{\lambda}(t) = & -\frac{\partial \mathcal{L}}{\partial x}(t, x(t), u(t))^T \\ & - \frac{\partial f}{\partial x}(t, x(t), u_n(t), u(r_u(t)))^T \lambda(t)\end{aligned} \tag{20}$$

$$\lambda(T) = 0,\ \nu(T) = 0 \tag{21}$$

$$\begin{aligned}\dot{\nu}(t) = & -\frac{\partial \mathcal{L}}{\partial u}(t, x(t), u(t))^T \\ & - \frac{\partial f}{\partial u}(t, x(t), u(t), u(r_u(t)))^T \lambda(t) \\ & - \mathbb{1}_{[0; r_u(T)]}(t) (r_u^{-1})'(t) \\ & \quad \cdot \frac{\partial f}{\partial u_r}(r_u^{-1}(t), x(r_u^{-1}(t)), \\ & \qquad u(r_u^{-1}(t)), u(t))^T \cdot \lambda(r_u^{-1}(t)) \\ & - \int_t^{r_u^{-1}(\min(t, r_u(T)))} \lambda(\tau)^T \\ & \quad \cdot \frac{\partial f}{\partial u_r}(\tau, x(\tau), u(\tau), u(r_u(\tau))) \\ & \qquad \cdot \frac{v(r_u(\tau))}{\phi(u(r_u(\tau)))}\, \mathrm{d}\tau \frac{\partial \phi}{\partial u}(u(t)).\end{aligned} \tag{22}$$

Using these notations, the stationarity conditions of $\mathcal{P}$ are given by

$$(u, x, \lambda, \nu) = \mathfrak{P}(v), \quad Pv + \nu = 0.$$

Solving $\mathcal{P}$ directly is difficult. Defining $(u_n, x_n, \lambda_n, \nu_n) \triangleq \mathfrak{P}(v_n)$ we would rather solve a sequence of simpler auxiliary problems $(\mathcal{P}_n)$, such that for all $n \geq 1$, $\mathcal{P}_{n+1}$ is defined according to (23) and (24) shown at the bottom of this page, is one of the two extra terms highlighted in Section II-C and is the sensitivity of the objective w.r.t. the change of the delay law caused by a change of the control input as derived from the calculus of variations.

It will become apparent from the subsequent analysis that if the sequence $(\mathcal{P}_n)$ admits a fixed point, it is necessarily a solution of Proposition 3. Introducing $\mathcal{S}_n$ term allows us to recover an unbiased approximation of the solution to the original problem $\mathcal{P}$ by solving $(\mathcal{P}_n)$. On the other hand, the problems $\mathcal{P}_n$ are much easier to solve than $\mathcal{P}$ because the delay law is fixed *a priori* and one can easily apply powerful existing optimization techniques, such as direct collocations, on these intermediate problems. Based on this insight, our goal throughout the rest of the discussion is to establish the conditions for the convergence of this sequence. The following assumptions are considered.

Assumption 1: L is twice continuously differentiable while f, ϕ are continuously differentiable, and there exists $K \geq 0$ such that $\forall (t, x, u) \in [0;T] \times \mathbb{R}^m \times \mathbb{R}^p$, $\|\nabla^2 L(t, x, u)\|_1 \leq K$ and $\forall (t, x, u, u_r) \in [0;T] \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p$, $\|\nabla f(t, x, u, u_r)\|_1 \leq K$ and $\forall u \in \mathbb{R}^p$, $\|\nabla \phi(u)\|_1 \leq K$ and, $\nabla^2 L, \nabla f, \nabla \phi$ are K-Lipschitz continuous.

Assumption 2: There exists $J^* \in \mathbb{R}$ such that $\forall v \in L^2([0;T])$, $J^* \leq J(v)$.

Assumption 3: There exists $\phi_{\min} > 0$ such that $\forall u \in \mathbb{R}$, $\phi_{\min} \leq \phi(u)$.

Remark 3: Assumptions 1 and 2 are classically considered in the optimization literature. Assumption 3 is usually considered for systems with input-varying delays of hydraulic type [26] so that r_u' be bounded away from zero and the input keeps on reaching the plant.

Definition 1: Given $\alpha \geq 0$, a sequence $(v_n)_{n \in \mathbb{N}^*}$ is called α-admissible if for all $n \geq 2$, v_n is a solution (possibly local) of $\mathcal{P}_n$.

Let us define

$$\begin{aligned}\mathcal{X} \triangleq \{ & v \in L^2([0;T]),\ \exists R_v \in \mathbb{R}_+,\ \forall w \in L^2([0;T]), \\ & J(w) \leq J(v) \Rightarrow \|w\|_2 \leq R_v \}\end{aligned} \tag{25}$$

the set of L^2 functions such that their J-level set is included in a ball of L^2 and note $g_v \triangleq Pv + \nu$. The main result concerning the sequence $(\mathcal{P}_n)$ is as follows.

$$\begin{aligned}\mathcal{P}_{n+1} : \min_{v_{n+1}} \int_0^T & \mathcal{L}(t, X_{n+1}(t), u_{n+1}(t)) + \frac{1}{2} v_{n+1}(t)^T P v_{n+1}(t) + \mathcal{S}_n(t)(u_{n+1}(t) - u_n(t)) + \frac{\alpha}{2} \|v_{n+1}(t) - v_n(t)\|_2^2\, \mathrm{d}t \\ \text{s.t.}\quad & \dot{X}_{n+1} = f(t, X_{n+1}(t), u_{n+1}(t), u_{n+1}(r_{u_n}(t))) \\ & \dot{u}_{n+1} = v_{n+1} \\ & X_{n+1}(0) = x_0,\ u_{n+1[r_0;0]} = u_0,\ v_{n+1[r_u(0);0]} = v_0\end{aligned} \tag{23}$$

$$\mathcal{S}_n(t) = \int_t^{r_{u_n}^{-1}(\min(t, r_{u_n}(T)))} \lambda_n(\tau)^T \frac{\partial f}{\partial u}(\tau, x_n(\tau), u_n(\tau), u_n(r_{u_n}(\tau))) \frac{v_n(r_{u_n}(\tau))}{\phi(u_n(r_{u_n}(\tau)))}\, \mathrm{d}\tau \frac{\partial \phi}{\partial u}(u_n(t)). \tag{24}$$

Theorem 2: Under Assumptions 1, 2 and 3, given any α-admissible sequence $(v_n)_{n\in\mathbb{N}^*}$ such that $v_1 \in \mathcal{X}$, if α is large enough then (v_n) satisfies $\lim_{n\to\infty} \|g_{v_n}\|_2 = 0$ and $\lim_{n\to\infty} \|v_{n+1} - v_n\|_2 = 0$.

Proof: Given $n \in \mathbb{N}^*$, let us assume that $v_n \in \mathcal{X}$ (which is true for $n = 1$ by assumption) and, by extension of (25), define $\mathcal{X}_n \triangleq \{v \in L^2([0;T]),\ J(v) \le J(v_n)\} \subset \mathcal{X}$, which is a bounded set in the sense of the L^2 norm, i.e., there exists $R_n > 0$ such that $\forall v \in \mathcal{X}_n,\ \|v\|_2 \le R_n$. Consider $\mathfrak{Q} : L^2([0;T], \mathbb{R}^p)^2 \to D^1([0;T], \mathbb{R}^p)^2 \times D^1([0;T], \mathbb{R}^m)^2$ with $\mathfrak{Q}(v, w) = (u, q, x, \lambda)$ defined as

$$\dot{u}(t) = v(t),\ u_{[r_0;0]} = u_0$$

$$\dot{q}(t) = w(t),\ q_{[r_0;0]} = u_0 \tag{26}$$

$$\dot{x}(t) = f(t, x(t), u(t), u(r_q(t))),\ x(0) = x_0 \tag{27}$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{L}}{\partial x}(t, x(t), u(t))^T - \frac{\partial f}{\partial x}(t, x(t), u(t), u(r_q(t)))^T \lambda(t),\ \lambda(T) = 0. \tag{28}$$

Note the slight (but important) differences between $\mathfrak{P}$ defined by (18)–(22) and $\mathfrak{Q}$. The second argument of $\mathfrak{Q}$ is used to define the time-varying delay appearing in the right-hand side of (27) and (28). Based on Assumption 1 and Cauchy existence and uniqueness theorem, $\mathfrak{Q}$ is clearly defined. Given the couples of arguments (v_1, w_1), (v_2, w_2) and (v, w), we define $(u_1, q_1, x_1, \lambda_1) = \mathfrak{Q}(v_1, w_1)$, $r_1 \triangleq r_{q_1}$, $(u_2, q_2, x_2, \lambda_2) = \mathfrak{Q}(v_2, w_2)$, $r_2 \triangleq r_{q_2}$ and $(u, q, x, \lambda) = \mathfrak{Q}(v, w)$, $r \triangleq r_q$ which are used to formulate the subsequent lemma.

Lemma 1 (Lipschitz continuity of $\mathfrak{Q}$): The two following inequalities hold, for all $t \in [0;T]$:

$$\|u_2(t) - u_1(t)\|_1 \le \sqrt{pt}\|v_2 - v_1\|_2 \tag{29}$$

$$\|u(t) - u_0\|_1 \le \sqrt{pt}\|v\|_2. \tag{30}$$

There exists some $(k_1, k_2, k_3, k_4) > 0$, $(l_1, l_2, l_3, l_4) > 0$ independent of α such that, for all $t \in [0;T]$

$$\|x_2(t) - x_1(t)\|_1 \le k_1\|v_2 - v_1\|_2 + k_2(1 + \|v_1\|_2) \cdot (1 + \|w_1\|_2 + \|w_2\|_2)\|w_2 - w_1\|_2 \tag{31}$$

$$\|x(t) - x_0\|_1 \le k_3 + k_4\|v\|_2$$

$$\|\lambda_2(t) - \lambda_1(t)\|_1 \le l_1(1 + \|v_1\|_2)\|v_2 - v_1\|_2 + l_2(1 + \|w_1\|_2 + \|w_2\|_2)(1 + \|v_1\|_2)^2\|w_2 - w_1\|_2 \tag{32}$$

$$\|\lambda(t)\|_1 \le l_3 + l_4\|v\|_2. \tag{33}$$

Proof: See Appendix A. ∎

The newly defined operator $\mathfrak{Q}$ plays a key role w.r.t. the sequence (v_n). Indeed, the stationarity conditions of $\mathcal{P}_{n+1}$ are given by

$$(u_{n+1}, X_{n+1}, \Lambda_{n+1}) = \mathfrak{Q}(v_{n+1}, v_n)$$

$$\dot{N}_{n+1}(t) = -\frac{\partial L}{\partial u}(t, X_{n+1}(t), u_{n+1}(t))^T - \frac{\partial f}{\partial u}(t, X_{n+1}(t), u_{n+1}(t), u_{n+1}(r_{u_n}(t)))^T \Lambda_{n+1}(t) - \mathbb{1}_{[0;r_{u_n}(T)]}(t)(r_{u_n}^{-1})'(t) \cdot \frac{\partial f}{\partial u_r}(r_{u_n}^{-1}(t), X_{n+1}(r_{u_n}^{-1}(t)), u_{n+1}(r_{u_n}^{-1}(t)), u_{n+1}(t))^T \cdot \Lambda_{n+1}(r_{u_n}^{-1}(t)) - \mathcal{S}_n(t)^T$$

$$0 = Pv_{n+1} + N_{n+1} + \alpha(v_{n+1} - v_n),\ N_{n+1}(T) = 0. \tag{34}$$

From this, we directly deduce that the solutions of $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ are related by

$$v_{n+1} = v_n - \frac{1}{\alpha}g_{v_n} + \frac{1}{\alpha}\epsilon_{n+1} \tag{35}$$

with $\epsilon_{n+1} = -P(v_{n+1} - v_n) - (N_{n+1} - \nu_n)$. In turn, the cost variation between v_n and v_{n+1} is given by $J(v_{n+1}) - J(v_n) = \int_0^1 G'(s)\,\mathrm{d}s$ where $G(s) = J(v_n + (v_{n+1} - v_n)s)$. Using the adjoint state method (e.g., [47]), one computes, after a few lines of calculus, $J(v_{n+1}) - J(v_n) = \int_0^1\int_0^T g_{v_n+(v_{n+1}-v_n)s}(t)^T(v_{n+1}(t) - v_n(t))\,\mathrm{d}t\,\mathrm{d}s$, which gives $J(v_{n+1}) - J(v_n) = -\frac{1}{\alpha}\|g_{v_n}\|_2^2 + \frac{1}{\alpha}\langle g_{v_n}, \epsilon_{n+1}\rangle + \int_0^1 \langle g_{v_n+(v_{n+1}-v_n)s} - g_{v_n}, v_{n+1} - v_n\rangle\,\mathrm{d}s$. Finally

$$J(v_{n+1}) - J(v_n) \le -\frac{1}{\alpha}\|g_{v_n}\|_2^2 + \frac{1}{\alpha}\|g_{v_n}\|_2\|\epsilon_{n+1}\|_2 + \int_0^1 \|g_{v_n+(v_{n+1}-v_n)s} - g_{v_n}\|_2\|v_{n+1} - v_n\|_2\,\mathrm{d}s. \tag{36}$$

To go further into the convergence analysis, we need to establish a bound for $\|\epsilon_{n+1}\|_2$ given in the following proposition.

Proposition 4: There exists some $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) > 0$ independent of α such that, for all $t \in [0;T]$

$$\|N_{n+1}(t) - \nu_n(t)\|_1 \le (\kappa_1 + \kappa_2\|v_n\|_2)\|v_{n+1} - v_n\|_2 \tag{37}$$

and

$$\|\nu_n(t)\|_1 \le \kappa_3 + \kappa_4\|v_n\|_2. \tag{38}$$

Proof: See Appendix B. ∎

Recalling (35), we have $\|v_{n+1} - v_n\|_2 \le \frac{1}{\alpha}(\|g_{v_n}\|_2 + \|P\|_2\|v_{n+1} - v_n\|_2 + \|N_{n+1} - \nu_n\|_2)$. Then, using (37) $\|v_{n+1} - v_n\|_2 \le \frac{1}{\alpha}\|g_{v_n}\|_2 + \frac{1}{\alpha}(\|P\|_2 + \kappa_1 + \kappa_2\|v_n\|_2)\|v_{n+1} - v_n\|_2$ As a consequence, if

$$\|P\|_2 + \kappa_1 + \kappa_2 R_n < \alpha \tag{39}$$

(which is always possible for α large enough as the left-hand side of (39) is independent of α), we find that

$$\|v_{n+1} - v_n\|_2 \le \frac{1}{\alpha - \|P\|_2 - \kappa_1 - \kappa_2 R_n}\|g_{v_n}\|_2. \tag{40}$$

In particular, we deduce an *a priori* bound on the norm of v_{n+1}, using (38) $\|v_{n+1}\|_2 \le R_n + \frac{\|P\|_2 R_n + \kappa_3 + \kappa_4 R_n}{\alpha - \|P\|_2 - \kappa_1 - \kappa_2 R_n}$. Incidentally, this also leads to

$$\|\epsilon_{n+1}\|_2 \le \frac{\|P\|_2 + \kappa_1 + \kappa_2 R_n}{\alpha - \|P\|_2 - \kappa_1 - \kappa_2 R_n}\|g_{v_n}\|_2. \tag{41}$$

To go further into the analysis of (36), we now have to prove the Lipschitz continuity of $g_v = Pv + \nu$ w.r.t. v. To do this, consider (v_1, v_2) and the associated functions $(u_1, x_1, \lambda_1, \nu_1) \triangleq \mathfrak{P}(v_1)$ and $(u_2, x_2, \lambda_2, \nu_2) \triangleq \mathfrak{P}(v_2)$.

Proposition 5: There exists a continuous function $\mathcal{K} : \mathbb{R}_+^3 \to \mathbb{R}_+$ increasing with each of its arguments and independent of α such that, for all $t \in [0;T]$

$$\|\nu_2(t) - \nu_1(t)\|_1 \leq \mathcal{K}(\|v_0\|_2, \|v_1\|_2, \|v_2\|_2)\, \|v_2 - v_1\|_2. \quad (42)$$

Proof: See Appendix C. ∎

One can then investigate further the decrease of cost formulated in (36). Using (41) and (42), ones gets $J(v_{n+1}) - J(v_n) \leq -\frac{1}{\alpha}(1 - \frac{\|P\|_2+\kappa_1+\kappa_2 R_n}{\alpha-\|P\|_2-\kappa_1-\kappa_2 R_n})\|g_{v_n}\|_2^2 + \frac{\mathcal{F}_n(\alpha)}{2}\ \|v_{n+1} - v_n\|_2^2$, where

$$\mathcal{F}_n(\alpha) = \mathcal{K}(\|v_0\|_2, R_n, R_n + \frac{\|P\|_2 R_n + \kappa_3 + \kappa_4 R_n}{\alpha - \|P\|_2 - \kappa_1 - \kappa_2 R_n}) + \|P\|_2.$$

Then, using (40) $J(v_{n+1}) - J(v_n) \leq -\frac{1}{\alpha}(1 - \frac{\|P\|_2+\kappa_1+\kappa_2 R_n}{\alpha-\|P\|_2-\kappa_1-\kappa_2 R_n} - \frac{\alpha \mathcal{F}_n(\alpha)}{2(\alpha-\|P\|_2-\kappa_1-\kappa_2 R_n)^2})\|g_{v_n}\|_2^2$. Since $\mathcal{F}_n$ is a decreasing function of α, there exists a value of α large enough such that

$$C(\alpha, R_n) \triangleq 1 - \frac{\|P\|_2 + \kappa_1 + \kappa_2 R_n}{\alpha - \|P\|_2 - \kappa_1 - \kappa_2 R_n} - \frac{\alpha \mathcal{F}_n(\alpha)}{2(\alpha - \|P\|_2 - \kappa_1 - \kappa_2 R_n)^2} > 0 \quad (43)$$

and $J(v_{n+1}) - J(v_n) < 0$. In particular, this guarantees that $v_{n+1} \in \mathcal{X}_n$. By induction, this implies that if one picks a value $\alpha = \alpha_1$ such that (α_1, R_1) satisfy (39) and (43), then for all rank n, $v_n \in \mathcal{X}_1$ and (39) and (43) hold. Then, $\forall n \in \mathbb{N}^*$, $J(v_{n+1}) - J(v_n) \leq -\frac{C(\alpha_1,R_1)}{\alpha_1}\|g_{v_n}\|_2^2$. This leads to $\sum_{i=0}^N \|g_{v_i}\|_2^2 \leq \frac{\alpha_1}{C(\alpha_1,R_1)}(J(v_0) - J(v_{n+1}))$. Finally, we derive $\sum_{i=0}^N \|g_{v_i}\|_2^2 \leq \frac{\alpha_1}{C(\alpha_1,R_1)}(J(v_0) - J^*)$ and $\lim_{n\to\infty} \|g_{v_n}\|_2 = 0$ which concludes the proof. ∎

IV. Numerical Example

We now illustrate the solution method studied in Theorem 2 using an example studied in [32]. Consider a second-order unstable linear system with dynamics given by $\ddot{z}(t) - \dot{z}(t) + z(t) = u(r_u(t))$, $\dot{u}(t) = v(t)$ having the following initial conditions: $z(0) = 1$, $\dot{z}(0) = 0$, $u_{[r_0;0]} = 1$, $v_{[r_0;0]} = 0$, and $\int_{r_u(t)}^t u(\tau)\,\mathrm{d}\tau = 1$. This can equivalently be recast as $\dot{X}(t) = AX(t) + Bu(r_u(t))$, $\dot{u}(t) = v(t)$, where $X = \left(\begin{smallmatrix} z \\ \dot{z}\end{smallmatrix}\right)$, $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1\end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right)$. Here, we seek to demonstrate the results of our approach by achieving the tracking of a time-varying reference. The optimal control problem is

$$\mathcal{P} : \min_v \int_0^T \|z(t) - z_r(t)\|_2^2 + w_v\|v(t)\|_2^2\, \mathrm{d}t$$
$$\text{s.t.}\quad \dot{X}(t) = AX(t) + Bu(r_u(t)),\ \dot{u}(t) = v(t)$$

with $T = 10$, $w_v = 0.01$, and $z_r(t) = 1 + \frac{1}{2}\sin(\frac{\pi}{4}\max(t - 2, 0))$. Given $\alpha = 5$, we approach iteratively a solution of $\mathcal{P}$ by constructing an α-admissible sequence[2] (v_n). We pick the trivial initialization value $v_1 = 0$ and for all $n \geq 1$ apply the algorithm whose flowchart is presented in Fig. 2. *First*, given v_n, compute u_n, and the delay law r_n, *second*, compute (x_n, λ_n) and deduce $\mathcal{S}_n$, *third,* solve $\mathcal{P}_{n+1}$ and obtain v_{n+1}. The algorithm is terminated when the variation of cost between two iterations goes below a given tolerance. At each step, r_n can be derived from (1) (either directly solving the integral equation or integrating the associated delay differential equation), while (x_n, λ_n) are computed by solving $\tilde{\mathcal{P}}_n$ and retrieving its optimal primal and adjoint states

$$\min_{v=v_n} \int_0^T \|z(t) - z_r(t)\|_2^2 + w_v\|v(t)\|_2^2\, \mathrm{d}t$$
$$\text{s.t.}\quad \dot{X}(t) = AX(t) + Bu(r_{u_n}(t)),\ \dot{u}(t) = v(t).$$

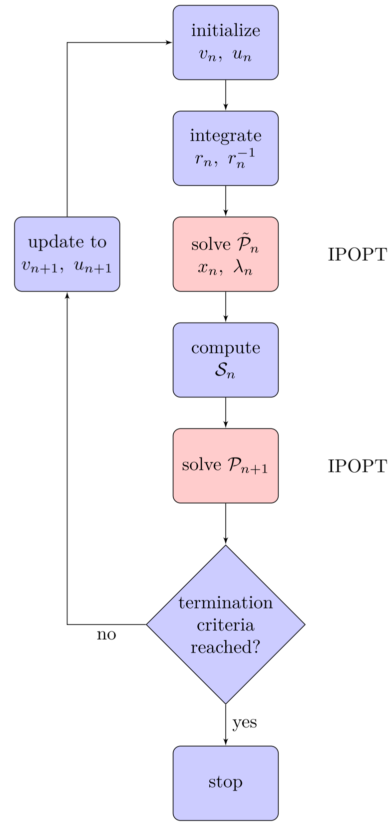
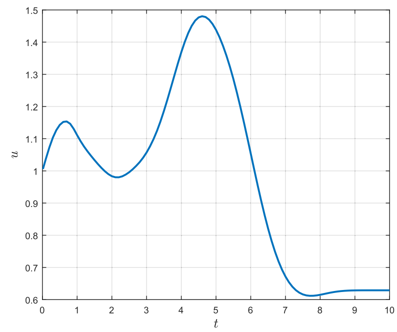


Fig. 2. Flowchart of the algorithm.

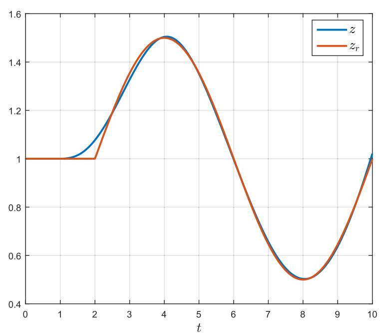
Practically, the resolution of $\mathcal{P}_{n+1}$ and $\tilde{\mathcal{P}}_n$ are performed using a direct collocation transcription method [48] with AMPL as algebraic modeling language and IPOPT 3.11.8 as NLP solver. The time horizon is divided into 100 finite elements of equal size, each of them containing three Radau collocation points. Figs. 3–5 report the optimal trajectory and the associated delay law.

[2]This value was chosen using a trial and error approach, knowing *a priori* that some large enough value of α would actually provide convergence.



Input

Fig. 3. Optimal trajectory computed for $\mathcal{P}$.



Output

Fig. 4. Optimal trajectory computed for $\mathcal{P}$.

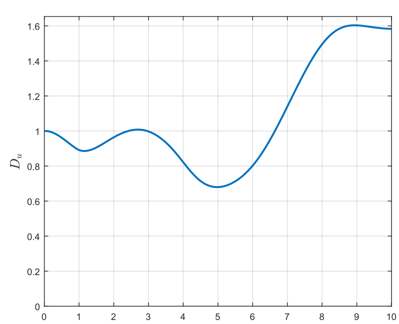


Fig. 5. Delay law of the optimal trajectory, as a function of time.

V. Conclusion

We carried the calculus of variations of the optimal control problem for a system subject to an input-dependent hydraulic delay. This establishes the noteworthy result that the straightforward formulation one could consider is ill-posed in the sense that it does not yield a smooth problem. On the other hand, fixed (i.e. not depending on the input) time-varying delays are much simpler to treat, and do not generate the discussed nondifferentiability in the Gâteaux sense. Following this, we introduced a regularization technique and an iterative algorithm, which only requires to solve a sequence of auxiliary problems with fixed time-varying delay laws, i.e., depending on the preceding input in the sequence. This allows us to use state-of-the-art optimization methods in the resolution of each auxiliary problem, while retrieving an unbiased solution. A convergence proof was detailed, showing that, similarly to a trust region method, our algorithm becomes equivalent to a gradient descent in the limit, where the allowed step-size goes to zero. Numerical results were given to illustrate the practical interest of the method.

An exciting problem raised by this article is the possibility of extending our approach to a second-order method. This would indeed greatly improve convergence performances in the neighborhood of the solutions. It is, however, not clear that the problem has sufficient regularity to directly allow such an extension. Establishing or disproving this would require the computation of the problem second variation.

Another straightforward development could be to extend the iterative optimization algorithm to the case of systems with hydraulic input-dependent state delays. This case is of importance since it is instrumental in the modeling of recycling loops or cascades of reacting units. The differentiability study presented here already covers these cases, but the numerical resolution algorithm does not.

This article has focused on the open-loop generation of optimal trajectories for the system. A valuable improvement would be to study the closed-loop behavior of such a methodology used in a receding horizon framework for real-time control applications. Stability conditions for such an MPC scheme could be productively investigated, following recent trends on the application of MPC to time-varying delays [49]–[52].

Appendix A
Proof of Lemma 1

Proof: Using Cauchy–Schwarz inequality

$$\begin{aligned}\|u_2(t)-u_1(t)\|_1 &= \left\| \int_0^t v_2(\tau)-v_1(\tau)\,\mathrm{d}\tau \right\|_1\\ &\leq \sqrt{pt}\|v_2-v_1\|_2 \end{aligned} \tag{44}$$

and similarly

$$\|u(t)-u_0(0)\|_1 \leq \sqrt{pt}\|v\|_2. \tag{45}$$

We also have

$$\begin{aligned}\|x_2(t)-x_1(t)\|_1 = \Bigg\| \int_0^t & f(\tau, x_2(\tau), u_2(\tau), u_2(r_2(\tau)))\\ & -f(\tau, x_1(\tau), u_1(\tau), u_1(r_1(\tau)))\,\mathrm{d}\tau \Bigg\|_1 .\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_2(t)-x_1(t)\|_1 \leq\\ &\int_0^t K\|x_2(\tau)-x_1(\tau)\|_1\,\mathrm{d}\tau + \int_0^t K\|u_2(\tau)-u_1(\tau)\|_1\,\mathrm{d}\tau\\ &+\int_0^t K\|u_2(r_2(\tau))-u_1(r_2(\tau))\|_1\\ &+\int_0^t K\|u_1(r_2(\tau))-u_1(r_1(\tau))\|_1\,\mathrm{d}\tau.\end{aligned}$$

Hence,

$$\|x_2(t) - x_1(t)\|_1 \leq K \int_0^t \|x_2(\tau) - x_1(\tau)\|_1 \,\mathrm{d}\tau + K \int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau + 2\, KT\sqrt{pT}\|v_2 - v_1\|_2. \quad (46)$$

Furthermore, we have

$$\int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau = \int_0^t \| \int_{r_1(\tau)}^{r_2(\tau)} v_1(s)\,\mathrm{d}s\|_1 \,\mathrm{d}\tau \leq \int_0^t \int_{a(\tau)}^{b(\tau)} \|v_1(s)\|_1 \,\mathrm{d}s\,\mathrm{d}\tau$$

where $a(s) \triangleq \min(r_1(s), r_2(s))$ and $b(s) \triangleq \max(r_1(s), r_2(s))$. Since r_1 and r_2 are strictly increasing functions, a and b also are invertible. From their respective definitions, it is also clear that $a(t) \leq b(t)$ and $a(0) = b(0) = r_0$. Then, using Fubini's theorem

$$\int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau \leq \int_{r_0}^{a(t)} \int_{b^{-1}(s)}^{a^{-1}(s)} \|v_1(s)\|_1 \,\mathrm{d}\tau\,\mathrm{d}s + \int_{a(t)}^{b(t)} \int_{b^{-1}(s)}^{t} \|v_1(s)\|_1 \,\mathrm{d}\tau\,\mathrm{d}s.$$

Hence

$$\int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau \leq \left(\sup_{s\in[r_0;a(t)]} (a^{-1}(s) - b^{-1}(s)) + \sup_{s\in[a(t);b(t)]} (t - b^{-1}(s)) \right) \cdot (\|v_1\|_1 + \|v_0\|_1)$$

where $\|v_0\|_1$ is used to denote $\|v_0\|_1 = \int_{r_0}^0 \|v_0(\tau)\|_1 \,\mathrm{d}\tau$ and similarly $\|v_0\|_2 = \sqrt{\int_{r_0}^0 \|v_0(\tau)\|_2^2 \,\mathrm{d}\tau}$. Then

$$\int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau \leq \left(\sup_{s\in[r_0;a(t)]} (a^{-1}(s) - b^{-1}(s)) + a^{-1}(a(t)) - b^{-1}(a(t)) \right) \cdot (\|v_1\|_1 + \|v_0\|_1).$$

For any $s \in [r_0; a(t)]$, $a^{-1}(s) - b^{-1}(s) = y_2 - y_1$ where y_1 and y_2 are uniquely defined by $s = a(y_2) = b(y_1)$. On the other hand, $\forall i \in \{1, 2\}$, using the Lipschitz continuity of ϕ, (17), (26), and integrating q either backward or forward, we find

$$r_i'(t) = \frac{\phi(q_i(t))}{\phi(q_i(r_i(t)))} \geq \frac{\phi_{\min}}{\phi(u_0(0)) + K(\sqrt{pT}\|w_i\|_2 + \sqrt{pr_0}\|v_0\|_2)}$$

a is a scalar function whose rate of change is lower bounded by the minimum of the two expressions of the previous equation. As a consequence

$$\frac{\phi_{\min} \cdot |y_2 - y_1|}{\phi(u_0(0)) + K(\sqrt{pT}(\|w_1\|_2 + \|w_2\|_2) + \sqrt{pr_0}\|v_0\|_2)} \leq |a(y_2) - a(y_1)|. \quad (47)$$

Then, since $|a(y_2) - a(y_1)| = |a(y_1) - b(y_1)| = |r_2(y_1) - r_1(y_1)|$, one has

$$\int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau \leq 2\frac{\phi(u_0(0)) + K(\sqrt{pT}(\|w_1\|_2 + \|w_2\|_2) + \sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}} \cdot \sup_{s\in[0;T]} |r_2(s) - r_1(s)| \cdot (\|v_1\|_1 + \|v_0\|_1).$$

Using (1), we have $\int_{r_1(t)}^t \phi(q_1(\tau))\,\mathrm{d}\tau - \int_{r_2(t)}^t \phi(q_2(\tau))\,\mathrm{d}\tau = 0$. Hence, $|\int_{r_2(t)}^{r_1(t)} \phi(q_2(\tau))\mathrm{d}\tau| \leq K \int_0^t \|q_2(\tau) - q_1(\tau)\|_1 \,\mathrm{d}\tau$ and then $|r_2(t) - r_1(t)| \leq \frac{Kt\sqrt{pT}}{\phi_{\min}}\|w_2 - w_1\|_2$ and

$$\int_0^t \|u_1(r_2(\tau)) - u_1(r_1(\tau))\|_1 \,\mathrm{d}\tau \leq 2\, KT\sqrt{pT} \cdot \frac{\phi(u_0(0)) + K(\sqrt{pT}(\|w_1\|_2 + \|w_2\|_2) + \sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}^2} \cdot (\|v_1\|_1 + \|v_0\|_1)\|w_2 - w_1\|_2. \quad (48)$$

Substituting in (46), this leads to

$$\|x_2(t) - x_1(t)\|_1 \leq K \int_0^t \|x_2(\tau) - x_1(\tau)\|_1 \,\mathrm{d}\tau + 2\, KT\sqrt{pT}\|v_2 - v_1\|_2 + \frac{\phi(u_0(0)) + K(\sqrt{pT}(\|w_1\|_2 + \|w_2\|_2) + \sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}^2} \cdot 2\, K^2\, T\sqrt{pT}(\|v_1\|_1 + \|v_0\|_1)\|w_2 - w_1\|_2.$$

Using Grönwall's lemma

$$\|x_2(t) - x_1(t)\|_1 \leq \left(\frac{\phi(u_0(0)) + K(\sqrt{pT}(\|w_1\|_2 + \|w_2\|_2) + \sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}^2} \cdot 2\, K^2\, T\sqrt{pT}(\|v_1\|_1 + \|v_0\|_1)\|w_2 - w_1\|_2 + 2\, KT\sqrt{pT}\|v_2 - v_1\|_2 \right) e^{Kt}.$$

Synthetically, a conservative estimate is as follows:

$$\|x_2(t) - x_1(t)\|_1 \leq k_1\|v_2 - v_1\|_2 + k_2(1 + \|w_1\|_2 + \|w_2\|_2)(1 + \|v_1\|_2)\|w_2 - w_1\|_2.$$

We also have for all $t \in [0;T]$,

$$\|x(t) - x_0\|_1 \le \int_0^t \|f(\tau, x(\tau), u(\tau), u(r_q(\tau)))\|_1 \,\mathrm{d}\tau.$$

Using the Lipschitz continuity of f

$$\|x(t) - x_0\|_1 \le \int_0^t \|f(0, x_0, u_0(0), u_0(r_0))\|_1 + K\tau + K\|x(\tau) - x_0\|_1 + K\|u(\tau) - u_0(0)\|_1 + K\|u(r(\tau)) - u_0(r_0)\|_1 \,\mathrm{d}\tau.$$

With Grönwall's lemma and (45), we find

$$\|x(t) - x_0\|_1 \le T(\|f(0, x_0, u_0(0), u_0(r_0))\|_1 + \frac{KT}{2} + 2\,K\sqrt{pT}\|v\|_2 + K\sqrt{pr_0}\|v_0\|_2)e^{KT}.$$

This is rewritten as

$$\|x(t) - x_0\|_1 \le k_3 + k_4\|v\|_2. \tag{49}$$

Let us define $\mu : t \mapsto \lambda(T-t)$. Then, integrating backward, one gets

$$\|\mu(t)\|_1 \le \int_0^t \|\frac{\partial L}{\partial x}(T-\tau, x(T-\tau), u(T-\tau)) + \frac{\partial f}{\partial x}(T-\tau, x(T-\tau), u(T-\tau), u(r_q(T-\tau)))^T \mu(\tau)\|_1 \,\mathrm{d}\tau.$$

Using the Lipschitz continuity of $\frac{\partial L}{\partial x}$, the boundedness of $\frac{\partial f}{\partial x}$, (44) and (49), we find

$$\|\mu(t)\|_1 \le K\int_0^t \|\mu(\tau)\|_1 \,\mathrm{d}\tau + T\left(\frac{\partial L}{\partial x}(0, x_0, u_0(0)) + KT + K(k_3 + k_4\|v\|_2) + K\sqrt{pT}\|v\|_2\right).$$

We deduce that the norm of the adjoint state is bounded

$$\|\lambda(t)\|_1 \le T\left(\frac{\partial L}{\partial x}(0, x_0, u_0) + KT + K(k_3 + k_4\|v\|_2) + K\sqrt{pT}\|v\|_2\right) e^{KT}$$

and

$$\|\lambda(t)\|_1 \le l_3 + l_4\|v\|_2. \tag{50}$$

We also have

$$\begin{aligned}&\|\mu_2(t) - \mu_1(t)\|_1 \\ &\le \int_0^t \left\| \frac{\partial L}{\partial x}(T-\tau, x_2(T-\tau), u_2(T-\tau)) - \frac{\partial L}{\partial x}(T-\tau, x_1(T-\tau), u_1(T-\tau))\right\|_1 \mathrm{d}\tau \\ &+ \int_0^t \left\| \frac{\partial f}{\partial x}(T-\tau, x_2(T-\tau), u_2(T-\tau), u_2(r_2(T-\tau)))^T \cdot \mu_2(\tau) - \frac{\partial f}{\partial x}(T-\tau, x_1(T-\tau), u_1(T-\tau) \right. \\ &\left. u_1(r_1(T-\tau)))^T \cdot \mu_1(\tau)\right\|_1 \mathrm{d}\tau.\end{aligned}$$

Consequently

$$\begin{aligned}&\|\mu_2(t) - \mu_1(t)\|_1 \\ &\le K\int_0^t \|x_2(T-\tau) - x_1(T-\tau)\|_1 + \|u_2(T-\tau) - u_1(T-\tau)\|_1 \,\mathrm{d}\tau \\ &+ \int_0^t K\|\mu_2(\tau) - \mu_1(\tau)\|_1 \\ &+ \left\|\left(\frac{\partial f}{\partial x}(T-t, x_2(T-\tau), u_2(T-\tau), u_2(r_2(T-\tau)))^T - \frac{\partial f}{\partial x}(T-t, x_1(T-\tau), u_1(T-\tau), u_1(r_1(T-\tau)))^T\right) \cdot \mu_1(\tau)\right\|_1 \mathrm{d}\tau\ .\end{aligned}$$

Then

$$\begin{aligned}&\|\mu_2(t) - \mu_1(t)\|_1 \\ &\le K(1 + l_3 + l_4\|v_1\|_2) \cdot \int_0^t \|x_2(T-\tau) - x_1(T-\tau)\|_1 + \|u_2(T-\tau) - u_1(T-\tau)\|_1 \,\mathrm{d}\tau \\ &+ K\int_0^t \|\mu_2(\tau) - \mu_1(\tau)\|_1 + \|u_2(r_2(T-t)) - u_1(r_1(T-t))\|_1\|\mu_1(\tau)\|_1 \,\mathrm{d}\tau.\end{aligned}$$

And, reusing (48), we obtain

$$\begin{aligned}&\|\mu_2(t) - \mu_1(t)\|_1 \le K(1 + l_3 + l_4\|v_1\|_2) \\ &\cdot \int_0^t k_1\|v_2 - v_1\|_2 + k_2(1 + \|w_1\|_2 + \|w_2\|_2) \cdot (1 + \|v_1\|_2)\|w_2 - w_1\|_2 + \sqrt{pT}\|v_2 - v_1\|_2 \,\mathrm{d}\tau \\ &+ K\int_0^t \|\mu_2(\tau) - \mu_1(\tau)\|_1 + \sqrt{pT}\|v_2 - v_1\|_2 \,\mathrm{d}\tau \\ &+ (l_3 + l_4\|v_1\|_2)2\,K^2\,T\sqrt{pT} \\ &\cdot \frac{\phi(u_0) + K(\sqrt{pT}(\|w_1\|_2 + \|w_2\|_2) + \sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}^2} \\ &\cdot (\|v_1\|_1 + \|v_0\|_1)\|w_2 - w_1\|_2.\end{aligned}$$

Again, using Grönwall's lemma, one finds

$$\begin{aligned}\|\lambda_2(t) - \lambda_1(t)\|_1 &\le l_1(1 + \|v_1\|_2)\|v_2 - v_1\|_2 \\ &+ l_2(1 + \|w_1\|_2 + \|w_2\|_2)(1 + \|v_1\|_2)^2\|w_2 - w_1\|_2.\end{aligned}$$

■

Appendix B
Proof of Proposition 4

Proof: We have

$$\begin{aligned}
&\|N_{n+1}(t)-\nu_n(t)\|_1\\
&\leq \int_t^T \left\|\frac{\partial L}{\partial u}(\tau, X_{n+1}(\tau), u_{n+1}(\tau)) - \frac{\partial L}{\partial u}(\tau, x_n(\tau), u_n(\tau))\right\|_1\\
&+ \left\|\frac{\partial f}{\partial u}(\tau, X_{n+1}(\tau), u_{n+1}(\tau), u_{n+1}(r_n(\tau)))^T \Lambda_{n+1}(\tau)\right.\\
&\left. - \frac{\partial f}{\partial u}(\tau, x_n(\tau), u_n(\tau), u_n(r_n(\tau)))^T \lambda_n(\tau)\right\|_1 \mathrm{d}\tau\\
&+ \int_{\min(t, r_n(T))}^{r_n(T)} \left\| \frac{\partial f}{\partial u_r}(r_n^{-1}(\tau), X_{n+1}(r_n^{-1}(\tau)),\right.\\
&\qquad u_{n+1}(r_n^{-1}(\tau)), u_{n+1}(\tau))\Lambda_{n+1}(r_n^{-1}(\tau))\\
&- \frac{\partial f}{\partial u_r}(r_n^{-1}(\tau), x_n(r_n^{-1}(\tau)),\\
&\qquad \left. u_n(r_n^{-1}(\tau)), u_n(\tau))\lambda_n(r_n^{-1}(\tau))\right\|_1 (r_n^{-1})'(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Using a change of variables in the second integral, we find

$$\begin{aligned}
&\|N_{n+1}(t)-\nu_n(t)\|_1\\
&\leq K\int_t^T \|X_{n+1}(\tau)-x_n(\tau)\|_1 + \|u_{n+1}(\tau)-u_n(\tau)\|_1\\
&\quad + \|\Lambda_{n+1}(\tau)-\lambda_n(\tau)\|_1\\
&\quad + \|\lambda_n(\tau)\|_1\,(\|X_{n+1}(\tau)-x_n(\tau)\|_1\\
&\quad + \|u_{n+1}(\tau)-u_n(\tau)\|_1\\
&\quad + \|u_{n+1}(r_n(\tau))-u_n(r_n(\tau))\|_1)\ \mathrm{d}\tau\\
&\quad + K\int_{r_n^{-1}(\min(t, r_n(T)))}^T \|\Lambda_{n+1}(\tau)-\lambda_n(\tau)\|_1\\
&\quad + \|\lambda_n(\tau)\|_1\,(\|X_{n+1}(\tau)-x_n(\tau)\|_1\\
&\quad + \|u_{n+1}(\tau)-u_n(\tau)\|_1\\
&\quad + \|u_{n+1}(r_n(\tau))-u_n(r_n(\tau))\|_1)\ \mathrm{d}\tau.
\end{aligned}$$

Finally, using the various Lipschitz continuity results established in Lemma 1, we find

$$\begin{aligned}
\|N_{n+1}(t)-\nu_n(t)\|_1 &\leq KT\Big(\sqrt{pT} + k_1 + 2l_1(1+\|v_n\|_2)\\
&+ 2(l_3 + l_3\|v_n\|_2)(2\sqrt{pT}+k_1)\Big)\,\|v_{n+1}-v_n\|_2.
\end{aligned}$$

This can be rewritten as $\|N_{n+1}(t)-\nu_n(t)\|_1 \leq (\kappa_1 + \kappa_2\|v_n\|_2)\|v_{n+1}-v_n\|_2$. It is also straightforward to show that, for some positive constants κ_3, κ_4, one has $\|\nu_n(t)\|_1 \leq \kappa_3 + \kappa_4\|v_n\|_2$. ■

Appendix C
Proof of Proposition 5

Proof: From (22), one has

$$\begin{aligned}
&\|\nu_2(t)-\nu_1(t)\|_1\\
&\leq \left\|\int_t^T \frac{\partial L}{\partial u}(\tau, x_2(\tau), u_2(\tau))^T - \frac{\partial L}{\partial u}(\tau, x_1(\tau), u_1(\tau))^T\right.\\
&\quad + \frac{\partial f}{\partial u}(\tau, x_2(\tau), u_2(\tau), u_2(r_2(\tau)))^T\lambda_2(\tau)\\
&\quad - \frac{\partial f}{\partial u}(\tau, x_1(\tau), u_1(\tau), u_1(r_1(\tau)))^T\lambda_1(\tau)\\
&\quad + \frac{\partial f}{\partial u_r}(r_2^{-1}(\tau), x_2(r_2^{-1}(\tau)), u_2(r_2^{-1}(\tau)), u_2(\tau))^T\\
&\quad\ \cdot \lambda_2(r_2^{-1}(\tau))\cdot \mathbb{1}_{[0;r_2(T)]}(\tau)(r_2^{-1})'(\tau)\\
&\quad - \frac{\partial f}{\partial u_r}(r_1^{-1}(\tau), x_1(r_1^{-1}(\tau)), u_1(r_1^{-1}(\tau)), u_1(\tau))^T\\
&\quad\ \cdot \lambda_1(r_1^{-1}(\tau))\cdot \mathbb{1}_{[0;r_1(T)]}(\tau)(r_1^{-1})'(\tau)\\
&\quad + \int_\tau^{r_2^{-1}(\min(\tau, r_2(T)))} \lambda_2(s)^T\\
&\qquad \cdot \frac{\partial f}{\partial u_r}(s, x_2(s), u_2(s), u_2(r_2(s)))\\
&\qquad \cdot \frac{v_2(r_2(s))}{\phi(u_2(r_2(s)))}\,\mathrm{d}s\,\frac{\partial \phi}{\partial u}(u_2(\tau))^T\\
&\quad - \int_\tau^{r_1^{-1}(\min(\tau, r_1(T)))} \lambda_1(s)^T\\
&\qquad \cdot \frac{\partial f}{\partial u_r}(s, x_1(s), u_1(s), u_1(r_1(s)))\\
&\qquad \left.\cdot \frac{v_1(r_1(s))}{\phi(u_1(r_1(s)))}\,\mathrm{d}s\,\frac{\partial \phi}{\partial u}(u_1(\tau))^T\,\mathrm{d}\tau\right\|_1.
\end{aligned}$$

Hence, using a change of variable in two of the integrals above, and after a Cauchy–Schwarz inequality, one gets

$$\begin{aligned}
&\|\nu_2(t)-\nu_1(t)\|_1\\
&\leq \left\|\int_t^T \frac{\partial L}{\partial u}(\tau, x_2(\tau), u_2(\tau))^T - \frac{\partial L}{\partial u}(\tau, x_1(\tau), u_1(\tau))^T\right.\\
&\quad + \frac{\partial f}{\partial u}(\tau, x_2(\tau), u_2(\tau), u_2(r_2(\tau)))^T\lambda_2(\tau)\\
&\quad \left.- \frac{\partial f}{\partial u}(\tau, x_1(\tau), u_1(\tau), u_1(r_1(\tau)))^T\lambda_1(\tau)\,\mathrm{d}\tau\right\|_1\\
&\quad + \left\|\int_{r_2^{-1}(\min(t, r_2(T)))}^T \frac{\partial f}{\partial u_r}(\tau, x_2(\tau), u_2(\tau), u_2(r_2(\tau)))^T\right.\\
&\quad\ \cdot \lambda_2(\tau)\,\mathrm{d}\tau\\
&\quad - \int_{r_1^{-1}(\min(t, r_1(T)))}^T \frac{\partial f}{\partial u_r}(\tau, x_1(\tau), u_1(\tau), u_1(r_1(\tau)))^T\\
&\quad\ \cdot \lambda_1(\tau)\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^T \int_\tau^{r_2^{-1}(\min(\tau,r_2(T)))} \lambda_2(s)^T \\
&\quad\cdot \frac{\partial f}{\partial u_r}(s,x_2(s),u_2(s),u_2(r_2(s))) \\
&\quad\cdot \frac{v_2(r_2(s))}{\phi(u_2(r_2(s)))}\,\mathrm{d}s \frac{\partial \phi}{\partial u}(u_2(\tau))^T \\
&-\int_\tau^{r_1^{-1}(\min(\tau,r_1(T)))} \lambda_1(s)^T \\
&\quad\cdot \frac{\partial f}{\partial u_r}(s,x_1(s),u_1(s),u_1(r_1(s))) \\
&\quad\cdot \frac{v_1(r_1(s))}{\phi(u_1(r_1(s)))}\,\mathrm{d}s \frac{\partial \phi}{\partial u}(u_1(\tau))^T \,\mathrm{d}\tau \Bigg\|_1 .
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\Bigg\| \int_{r_2^{-1}(\min(t,r_2(T)))}^T \frac{\partial f}{\partial u_r}(\tau,x_2(\tau),u_2(\tau),u_2(r_2(\tau)))^T \\
&\quad\cdot \lambda_2(\tau)\,\mathrm{d}\tau \\
&\quad-\int_{r_1^{-1}(\min(t,r_1(T)))}^T \frac{\partial f}{\partial u_r}(\tau,x_1(\tau),u_1(\tau),u_1(r_1(\tau)))^T \\
&\quad\cdot \lambda_1(\tau)\,\mathrm{d}\tau \Bigg\|_1 \\
&\le \int_t^T \Bigg\| \frac{\partial f}{\partial u_r}(\tau,x_2(\tau),u_2(\tau),u_2(r_2(\tau)))^T \lambda_2(\tau) \\
&\quad-\frac{\partial f}{\partial u_r}(\tau,x_1(\tau),u_1(\tau),u_1(r_1(\tau)))^T \lambda_1(\tau) \Bigg\|_1 \mathrm{d}\tau \\
&\quad+\Bigg\| \int_{r_1^{-1}(\min(t,r_1(T)))}^{r_2^{-1}(\min(t,r_2(T)))} \frac{\partial f}{\partial u_r}(\tau,x_1(\tau),u_1(\tau),u_1(r_1(\tau)))^T \\
&\quad\cdot \lambda_1(\tau)\,\mathrm{d}\tau \Bigg\|_1 .
\end{aligned}
$$

We get

$$
\begin{aligned}
&\|\nu_2(t)-\nu_1(t)\|_1 \\
&\le K(T-t)\sup_{\tau\in[0;T]} \left(\|x_2(\tau)-x_1(\tau)\|_1 + \|u_2(\tau)-u_1(\tau)\|_1\right. \\
&\quad\left.+\,2\,\|\lambda_2(\tau)-\lambda_1(\tau)\|_1\right) \\
&\quad+2K(T-t)(l_3+l_4\|v_1\|_2) \\
&\quad\cdot\Bigg(\sup_{\tau\in[0;T]}\left(\|x_2(\tau)-x_1(\tau)\|_1 + 2\|u_2(\tau)-u_1(\tau)\|_1\right) \\
&\qquad+\int_0^T \|u_1(r_2(t))-u_1(r_1(t))\|_1\,\mathrm{d}\tau\Bigg) \\
&+\Bigg\| \underbrace{\int_{r_1^{-1}(\min(t,r_1(T)))}^{r_2^{-1}(\min(t,r_2(T)))} \frac{\partial f}{\partial u_r}(\tau,x_1(\tau),u_1(\tau),u_1(r_1(\tau)))^T \cdot\lambda_1(\tau)\,\mathrm{d}\tau \Bigg\|_1}_{\triangleq A} \\
&+\underbrace{\Bigg\| \int_t^T \int_{r_1^{-1}(\min(\tau,r_1(T)))}^{r_2^{-1}(\min(\tau,r_2(T)))} \lambda_2(s)^T \cdot \frac{\partial f}{\partial u_r}(s,x_2(s),u_2(s),u_2(r_2(s))) \cdot\frac{v_2(r_2(s))}{\phi(u_2(r_2(s)))}\,\mathrm{d}s \frac{\partial \phi}{\partial u}(u_2(\tau))\,\mathrm{d}\tau \Bigg\|_1}_{\triangleq B} \\
&+\Bigg\| \int_t^T \int_\tau^{r_1^{-1}(\min(\tau,r_1(T)))} \lambda_2(s)^T \\
&\quad\cdot \frac{\partial f}{\partial u_r}(s,x_2(s),u_2(s),u_2(r_2(s))) \\
&\quad\cdot \frac{v_2(r_2(s))}{\phi(u_2(r_2(s)))}\,\mathrm{d}s \frac{\partial \phi}{\partial u}(u_2(\tau)) \\
&\quad-\lambda_1(s)^T \cdot \frac{\partial f}{\partial u_r}(s,x_1(s),u_1(s),u_1(r_1(s))) \\
&\quad\underbrace{\cdot\frac{v_1(r_1(s))}{\phi(u_1(r_1(s)))}\,\mathrm{d}s \frac{\partial \phi}{\partial u}(u_1(\tau))\,\mathrm{d}\tau \Bigg\|_1}_{\triangleq C} .
\end{aligned}
$$

We have

$$
A \le \int_{r_1^{-1}(\min(t,r_1(T)))}^{r_2^{-1}(\min(t,r_2(T)))} K(l_3+l_4\|v_1\|_2)\,\mathrm{d}\tau.
$$

The definition of the delay (1) gives us

$$
\begin{aligned}
&\int_{\min(t,r_2(T))}^{r_2^{-1}(\min(t,r_2(T)))} \phi(u_2(\tau))\,\mathrm{d}\tau \\
&\qquad= \int_{\min(t,r_1(T))}^{r_1^{-1}(\min(t,r_1(T)))} \phi(u_1(\tau))\,\mathrm{d}\tau.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\int_{r_1^{-1}(\min(t,r_1(T)))}^{r_2^{-1}(\min(t,r_2(T)))} \phi(u_2(\tau))\,\mathrm{d}\tau = \\
&-\int_{\min(t,r_1(T))}^{r_1^{-1}(\min(t,r_2(T)))} \phi(u_2(\tau))-\phi(u_1(\tau))\,\mathrm{d}\tau \\
&-\int_{\min(t,r_2(T))}^{\min(t,r_1(T))} \phi(u_2(\tau))\,\mathrm{d}\tau. \qquad (51)
\end{aligned}
$$

Moreover, (1) also implies

$$
\begin{aligned}
&\left|\int_{\min(t,r_2(T))}^{\min(t,r_1(T))} \phi(u_2(\tau))\,\mathrm{d}\tau\right| \le \left|\int_{r_2(T)}^{r_1(T)} \phi(u_2(\tau))\,\mathrm{d}\tau\right| \\
&\qquad= \left|\int_{r_1(T)}^T \phi(u_2(\tau))-\phi(u_1(\tau))\,\mathrm{d}\tau\right| \\
&\qquad\le K(T-r_0)\sqrt{pT}\|v_2-v_1\|_2.
\end{aligned}
$$

Then, using (51) and performing the same calculation

$$r_2^{-1}(\min(t, r_2(T))) - r_1^{-1}(\min(t, r_1(T))) \leq \frac{2\,K(T-r_0)\sqrt{pT}}{\phi_{\min}}\|v_2 - v_1\|_2.$$

Finally, we have

$$A \leq \frac{2\,K^2(T-r_0)\sqrt{pT}}{\phi_{\min}}(l_3 + l_4\|v_1\|_2)\|v_2 - v_1\|_2.$$

To treat B, we note

$$a:\ [r_0; \max(r_1(T), r_2(T))] \to [0;T]$$
$$t \mapsto \min\left(r_1^{-1}(\min(\tau, r_1(T))), r_2^{-1}(\min(\tau, r_2(T)))\right)$$

and

$$b:\ [r_0; \min(r_1(T), r_2(T))] \to [0;T]$$
$$t \mapsto \max\left(r_1^{-1}(\min(\tau, r_1(T))), r_2^{-1}(\min(\tau, r_2(T)))\right).$$

Since r_1^{-1} and r_2^{-1} are both strictly increasing functions, a and b both are invertible functions and

$$B \leq \int_{a(t)}^{b(t)}\int_t^{a^{-1}(s)} \left\|\lambda_2(s)^T \cdot \frac{\partial f}{\partial u_r}(s, x_2(s), u_2(s), u_2(r_2(s))) \cdot \frac{v_2(r_2(s))}{\phi(u_2(r_2(s)))}\frac{\partial \phi}{\partial u}(u_2(\tau))\right\|_1 \mathrm{d}\tau\,\mathrm{d}s$$
$$+ \int_{b(t)}^{T}\int_{b^{-1}(s)}^{a^{-1}(s)} \left\|\lambda_2(s)^T \cdot \frac{\partial f}{\partial u_r}(s, x_2(s), u_2(s), u_2(r_2(s))) \cdot \frac{v_2(r_2(s))}{\phi(u_2(r_2(s)))}\frac{\partial \phi}{\partial u}(u_2(\tau))\right\|_1 \mathrm{d}\tau\,\mathrm{d}s.$$

Then, by the Lipschitz continuity of $\frac{\partial \phi}{\partial u}$ and the boundedness of $\frac{\partial f}{\partial u_r}$

$$B \leq \left(\sup_{s\in[a(t);b(t)]}(a^{-1}(s) - t) + \sup_{s\in[b(t);T]}(a^{-1}(s) - b^{-1}(s))\right) \cdot (l_3 + l_4\|v_2\|_2)\frac{K^2}{\phi_{\min}}\int_0^T \|v_2(r_2(s))\|_1\,\mathrm{d}s.$$

Besides, by the Cauchy–Schwarz inequality

$$\int_0^T \|v_2(r_2(s))\|_1\,\mathrm{d}s \leq \sqrt{pT}\sqrt{\int_0^T \|v_2(r_2(s))\|_2^2\,\mathrm{d}s}$$
$$\leq \sqrt{pT}\sqrt{\int_{r_2(0)}^{r_2(T)} \|v_2(s)\|_2^2 \cdot (r_2^{-1})'(s)\,\mathrm{d}s}$$
$$\leq \sqrt{\frac{pT(\phi(u_0(0)) + K\sqrt{pT}\|v_2\|_2 + K\sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}}} \cdot (\|v_2\|_2 + \|v_0\|_2)$$

and since $a \leq b$

$$B \leq \left(a^{-1}(b(t)) - b^{-1}(b(t)) + \sup_{s\in[b(t);T]}(a^{-1}(s) - b^{-1}(s))\right) \cdot (l_3 + l_4\|v_2\|_2)\frac{K^2}{\phi_{\min}}\sqrt{\frac{pT(\phi(u_0(0)) + K\sqrt{pT}\|v_2\|_2)}{\phi_{\min}}} \cdot (\|v_2\|_2 + \|v_0\|_2).$$

Finally, after a few lines of calculus similar to (47), we get

$$B \leq 2(l_3 + l_4\|v_2\|_2)\frac{K^2}{\phi_{\min}} \cdot (\|v_2\|_2 + \|v_0\|_2)\,\|v_2 - v_1\|_2 \cdot \sqrt{\frac{pT(\phi(u_0(0)) + K\sqrt{pT}\|v_2\|_2) + K\sqrt{pr_0}\|v_0\|_2}{\phi_{\min}}} \cdot \frac{KT\sqrt{pT}(\phi(u_0(0)) + K\sqrt{pT}\|v_1\|_2 + K\sqrt{pr_0}\|v_0\|_2)}{\phi_{\min}^2}.$$

Using the same kind of computations on C, we show that, for all $t \in [0;T]$,

$$\|\nu_2(t) - \nu_1(t)\|_1 \leq \mathcal{K}(\|v_0\|_2, \|v_1\|_2, \|v_2\|_2)\,\|v_2 - v_1\|_2$$

where $\mathcal{K}: \mathbb{R}_+^3 \to \mathbb{R}_+$ is a continuous function such that for all i

$$x_i \leq z_i \Rightarrow \mathcal{K}(x_1, x_2, x_3) \leq \mathcal{K}(z_1, z_2, z_3)$$

which gives the conclusion. ■

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