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Robust compensation of a chattering time-varying input delay

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Abstract— **We investigate the design of a prediction-based controller for a linear system subject to a time-varying input delay, not necessarily causal. This means that the information feeding the system can be older than ones previously received. We propose to use the current delay value in the prediction employed in the control law. Modeling the input delay as a transport Partial Differential Equation, we prove asymptotic tracking of the system state, providing that the average $\mathscr{L}_2$-norm of the delay time-derivative is sufficiently small. This result is obtained by generalizing Halanay inequality to time-varying differential inequalities.**

I. Introduction

Time-delays are ubiquitous in engineering systems, which often involve either communication lags or a physical dead-time which reveals troublesome in the design and tuning of feedback control laws. The latter occurs, e.g., for processes including transportation of material, such as mixing plants for liquid or gaseous fluids [7] [24], automotive engine and exhaust line [8] or heat collector plant [26], among others. In all these examples, the dead-time is therefore a transport delay, which satisfies inherently a causality property. However, this does not hold in general, as, e.g., communication delays can be subject to sudden variations and therefore do not vary according to the "First-In-First-Out" principle. This non-causality phenomenon can also occur for input-dependent input delay systems [9], in which the delay variations can be related to the input in a very intricate manner, like, e.g., for crushing mill devices [25].

In this paper, we consider a time-varying input delay which can violate this causality principle. We investigate the design of a prediction-based control law [1] [17] [18] [27], which is state-of-the-art for constant input delays [5] [11] [14] [19] [21] [22] but is still not of general use for time-varying delays (see [23] or, more recently, [16]). In such cases, to compensate the varying input delay, the prediction has to be calculated over a time window of which length matches the value of the future delay. In other words, one may need to predict the future variations of the delay to compensate it. For example, this is the approach followed in [29] for a communication time-varying delay, the variations of which are provided by a given known model. It has also been used in [2] for a state-dependent delay or in [3] for a delay depending on delayed state, where variations are anticipated by a careful prediction of the system state. However, it may not be possible in general to compute such a time horizon if the delay is not causal, as discussed later in Section II.

For this reason, in this paper, in lieu of seeking exact delay compensation, we consider a prediction horizon equal to the current delay value, which is assumed to be known. The delay itself is not necessarily causal, i.e., it can be such that $\dot{D}(t) > 1$ for some $t \geq 0$. The meaning of this condition is that the delay can vary more rapidly that the absolute time for some instants. In other words, older information can temporally feed the system. Up to the authors' knowledge, this situation has never been studied, as all previous works consider that $\dot{D}(t) \leq 1$ for $t \geq 0$ (see [4] [10] [20] [30] for instance). This is the main contribution of the paper. As a first step in the design of prediction-based control law for systems subject to chattering input delay, we consider the delay function to be continuously differentiable, which is a demanding assumption from a practical point of view and should be relaxed in future works. Recasting the problem as an Ordinary Differential Equation (ODE) cascaded with a transport Partial Differential Equation (PDE), we use a backstepping transformation recently introduced in [15] to analyze the closed-loop stability. Extending Halanay inequality [6], [12], [13], [28] to the linear time-varying framework, we prove asymptotic convergence of the system state provided that the delay time-derivative is sufficiently small in average, in the sense of an average $\mathscr{L}_2$-norm.

The paper is organized as follows. In Section II, we introduce the problem at stake, before designing our control strategy and stating our main result. The latter is proved in Section III and its merits are illustrated in Section IV with a simulation example. We conclude with directions of future work.

Notations. In the following, $|\cdot|$ is the usual Euclidean norm and, for a signal $u(x,\cdot)$ for $x \in [0,1]$, $\|u(\cdot)\|$ denotes the spatial $\mathscr{L}_2$-norm., i.e.,

$$\|u(t)\| = \sqrt{\int_0^1 u(x,t)^2 dx}$$

We write $\partial_x f$ the partial derivative of a function f with respect to a variable x; x_t refers to the function $x_t : s \in [-\overline{D}, 0] \to x(t+s)$ for a given function x and a constant $\overline{D} > 0$. Finally, $\underline{\lambda}(M)$ and $\overline{\lambda}(M)$ refer to the minimal and maximal eigenvalues of a matrix M.

II. Problem statement and control design

We consider the following (potentially) unstable linear dynamics

$$\dot{X}(t) = AX(t) + BU(t - D(t)) \tag{1}$$

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in which the delay is a *known* continuously differentiable function such that $D(t) \in [\underline{D}, \overline{D}'] \subset [\underline{D}, \overline{D}]$ with $\underline{D} > 0$. Note that no assumption is made a priori on the time-derivative of D. In particular, it is possible that $\dot{D}(t) > 1$ for certain intervals of time.

The control objective is to design a prediction-based controller stabilizing the plant (1), taking advantage of the fact that the current value of the delay is known for all time. With this aim in view, consider the following control law

$$U(t) = K\left[e^{AD(t)}X(t) + \int_{t-D(t)}^{t} e^{A(t-s)}BU(s)ds\right] \qquad (2)$$

in which the feedback gain K is such that $A+BK$ is Hurwitz.

This controller aims at forecasting values of the state over a time window of varying length $D(t)$[1]. Of course, exact compensation of the delay is not achieved with this controller. To do so, one would need to consider a time window of which length would exactly match the value of the future delay, as it is made in [23] and [16]. In details, defining $\eta(t) = t - D(t)$ and assuming that its inverse exists, exact delay-compensation is obtained with the feedback law $U(t) = KX(\eta^{-1}(t))$. Yet, implementing this relation requires to predict the future variations of the delay via $\eta^{-1}(t)$. This may not be achieved in practice, when no delay model is available. Further, note that the inverse function $\eta^{-1}(t)$ may not exist for all time, if $\dot{D}(s) > 1$ for some instants as η may then be non-monotonically increasing. This motivates our choice of the prediction-based controller (2).

Theorem 1: Consider the closed-loop system consisting of the dynamics (1) and the control law (2) in which the delay $D : \mathbb{R} \to [\underline{D}, \overline{D}'] \subset [\underline{D}, \overline{D}]$ is a continuously differentiable function such that there exists $\delta > 0$ such that

$$\forall t \geq 0 \quad \frac{1}{t-h_i}\int_{h_i}^{t} \dot{D}(s)^2 ds \leq \delta\,, \quad t \in [h_i, h_{i+1}] \qquad (3)$$

for an ordered sequence $(h_i)_{i\in\mathbb{N}}$ such that $h_0 = t_0$, $\lim_{i\to\infty} h_i = \infty$ and $\Delta h_i = h_{i+1} - h_i \in [\underline{\Delta}, \overline{\Delta}]$ for $i \in \mathbb{N}$. Define the functional

$$\Gamma(t) = |X(t)|^2 + \int_{t-\overline{D}}^{t} U(s)^2 ds$$

There exists $\delta^* \in (0,1)$ such that, if $\delta < \delta^*$, there exist $\gamma, R > 0$ such that

$$\Gamma(t) \leq R \max \Gamma_{\overline{D}}\, e^{-\gamma(t-\overline{D})}\,,\ t \geq \overline{D} \qquad (4)$$

[1]Note that this controller does not exactly match the predicted system state on a time-horizon $D(t)$. Indeed, using the variation of constant formula

$$\forall t \geq 0\,, \quad X(t+D(t)) = e^{AD(t)}X(t) + \int_{t-D(t)}^{t} e^{A(t-s)}BU(s+D(t)-D(s))ds$$

However, the integral in this prediction may not be implementable as it is not necessarily causal (in details, this is the case when there exists $s \in [t-D(t), t]$ such that $s - D(s) \geq t - D(t)$, i.e., when the delay $D(t)$ is suddenly high and the signal received at time t is older then those previously received) while the one employed in (2) always is.

Further, even if one can implement this prediction, the involved integral can be approximated by the one used in (2) if $D(t) - D(s) \approx 0$ for "most" instant t, i.e., under the assumption that the variations of the delay are sufficiently small in average. As this assumption is the one which is required in the following in Theorem 1 to robustly compensate the delay, we rather use the prediction form (2) which is always causal and easier to implement.

Condition (3) allows the delay time-derivative to be quite large for some time instants, but requires it to be sufficiently small in average to guarantee stability, that is in the sense of the average $\mathscr{L}_2$-norm given in condition (3). In particular, the delay function can be non-causal for some time instants, as long as it is not most of the time (i.e. as $\delta^* < 1$).

Note that, as our prediction employs the current delay value $D(t)$ instead of the time horizon $\eta^{-1}(t)$ to estimate the future system state, it can be highly inaccurate when the delay is fast varying. In this context, the requirement $\delta < \delta^*$ with δ introduced in (3) can also be interpreted as a condition for robust delay compensation achievement: if the delay varies sufficiently slowly most of the time, its current value $D(t)$ used for prediction will remain sufficiently often close enough to its future values for the corresponding prediction to guarantee closed-loop stabilization.

We now detail the proof of this theorem.

III. Proof of Theorem 1

A. Backstepping transformation and target system

As a fist step in our analysis, we introduce the two distributed actuators

$$u(x,t) = U(t + D(t)(x-1)) \qquad (5)$$
$$v(x,t) = U(t - \overline{D} + x(\overline{D} - D(t))) \qquad (6)$$

to reformulate (1) into the following PDEs-ODE cascade

$$\dot{X}(t) = AX(t) + Bu(0,t) \qquad (7)$$
$$D(t)\partial_t u = (1 + \dot{D}(t)(x-1))\partial_x u \qquad (8)$$
$$u(1,t) = U(t) \qquad (9)$$
$$(\overline{D} - D(t))\partial_t v = (1 - x\dot{D}(t))\partial_x v \qquad (10)$$
$$v(1,t) = u(0,t) \qquad (11)$$

In details, the input delay is now represented as the cascade of an ODE (7) fed by the output of a transport PDE (8), with time- and space- varying propagation velocity. The first transport PDE (8) is cascaded with a second transport PDE (10) with space- and time-varying propagation velocity. Together, (8)–(11) simply account for the input propagation over a time window of length $\overline{D}$. Note that, as no assumption is made a priori on the existence of a upper-bound of the delay derivative, the pointwise velocities in (8) and (10) can be positive or negative depending on the spatial variable.

To analyze this closed-loop system, following [15], we define the following backstepping transformation

$$w(x,t) = u(x,t) - K\left[e^{AD(t)x}X(t) + D(t)\int_0^x e^{AD(t)(x-y)}Bu(y,t)dy\right] \qquad (12)$$

Lemma 1: The infinite-dimensional backstepping transformation (12) together with the control law (2) transform the plant (1) into the target system

$$\dot{X}(t) = (A+BK)X(t) + Bw(0,t) \qquad (13)$$
$$D(t)\partial_t w = (1 + \dot{D}(t)(x-1))\partial_x w - D(t)\dot{D}(t)g(x,t) \qquad (14)$$
$$w(1,t) = 0 \qquad (15)$$

$$(\overline{D}-D(t))\partial_t v = (1-x\dot{D}(t))\partial_x v \tag{16}$$
$$v(1,t) = u(0,t) \tag{17}$$

with $g(x,t) = Ke^{AD(t)x}(AX(t)+Bu(0,t))$

Proof: Taking time- and space-derivative of (12), one gets, using integration by parts for the second equation,

$$\partial_t w = \partial_t u - K\dot{D}(t)\Big[e^{AD(t)x}AxX(t) + \int_0^x e^{AD(t)(x-y)}(I+AD(t)(x-y))Bu(y,t)dy\Big] - K\Big[e^{AD(t)x}(AX+Bu(0,t)) + D(t)\int_0^x e^{AD(t)(x-y)}B\partial_t u(y,t)dy\Big]$$

$$\partial_x w = \partial_x u - K\Big[e^{AD(t)x}AD(t)X + D(t)e^{AD(t)x}Bu(0,t) + D(t)\int_0^x e^{AD(t)(x-y)}B\partial_x u(y,t)dy\Big]$$

Matching these two expressions and using (8), one easily gets (14) with

$$g(x,t) = K\Big[e^{AD(t)x}AxX(t) + \int_0^x e^{AD(t)(x-y)}(I+AD(t)(x-y))Bu(y,t)dy\Big] + K(1-x)\times\Big[e^{AD(t)x}(AX+Bu(0,t)) + \int_0^x e^{AD(t)(x-y)}B\partial_x u(y,t)dy\Big] + K\int_0^x e^{AD(t)(x-y)}B(y-1)\partial_x u(y,t)dy$$

which, using the integration by part

$$\int_0^x e^{AD(t)(x-y)}B(y-x)\partial_x u(y,t)dy = e^{AD(t)x}xBu(0,t) - \int_0^x e^{AD(t)(x-y)}(I+AD(t)(x-y))Bu(y,t)dy$$

can simply be expressed as in Lemma 1. The boundary condition (15) follows from the choice of the control law (2) and the backstepping transformation definition (12). ■

As the target system presents the suitable boundary condition $w(1,t)=0$, this the one which is used in the Lyapunov analysis.

B. Stability analysis

Consider the following Lyapunov functional candidate

$$V(t) = X(t)^T PX(t) + b_1 D(t)\int_0^1 (1+x)w(x,t)^2dx + b_2(\overline{D}-D(t))\int_0^1 (1+x)v(x,t)^2dx \tag{18}$$

in which P is the symmetric positive-definite solution of the Lyapunov equation $P(A+BK)+(A+BK)^TP=-Q$, for a given symmetric definite-positive matrix Q and b_1, b_2 are positive constant parameters. Note that, using Young and Cauchy-Schwartz inequalities, together with the inverse Backstepping transformation

$$u(x,t) = w(x,t) + K\Big[e^{(A+BK)D(t)}X(t) + D(t)\int_0^x e^{(A+BK)D(t)(x-y)}Bw(y,t)dy\Big] \tag{19}$$

one gets the existence of constants $r_1, r_2, s_1, s_2 > 0$ such that

$$\|u(t)\|^2 \le r_1|X(t)|^2 + r_2\|w(t)\|^2 \tag{20}$$
$$\|w(t)\|^2 \le s_1|X(t)|^2 + s_2\|u(t)\|^2 \tag{21}$$

and hence, observing that $\int_{t-D(t)}^t U(s)^2ds = D(t)\|u(t)\|^2$ and that $\int_{t-\overline{D}}^{t-D(t)} U(s)^2ds = (\overline{D}-D(t))\|v(t)\|^2$, one obtains the existence of $\mu_1, \mu_2 > 0$ such that

$$\mu_1\Gamma(t) \le V(t) \le \mu_2\Gamma(t) \tag{22}$$

Now, taking a time-derivative and using integrations by parts, one gets

$$\dot{V}(t) = -X^TQX + 2X^TPBw(0,t) + b_1\Big(-(1-\dot{D}(t))w(0,t)^2 - \|w(t)\|^2 - 2\dot{D}(t)\int_0^1 xw(x,t)^2dx - 2D(t)\dot{D}(t)\int_0^1 (1+x)w(x,t)g(x,t)dx\Big) + b_2\Big(2v(1,t)^2 - v(0,t)^2 - \|v(t)\|^2 - 2\dot{D}(t)v(1,t)^2 + \dot{D}(t)\int_0^1 (2x+1)v(x,t)^2dx\Big) + \dot{D}(t)\int_0^1 (1+x)[b_1w(x,t)^2 - b_2v(x,t)^2]dx$$

in which, from (11) and (12),

$$2v(1,t)^2 \le 4(w(0,t)^2 + |K|^2|X(t)|^2) \tag{23}$$

Using the fact that, from (2) with Young and Cauchy-Schwartz inequalities,

$$u(0,t)^2 = U(t-D(t))^2 \le \tilde{M}(|X(t-D(t))|^2 + \|u(t-D(t))\|^2),\ t \ge \overline{D} \tag{24}$$

for a given positive constant $\tilde{M}$, together with (20) and Young and Cauchy-Schwartz inequalities, one obtains the existence of a constant $M>0$ such that

$$\Big|2D(t)\int_0^1 (1+x)w(x,t)g(x,t)dx\Big| \le W(t) \tag{25}$$
$$2v(1,t)^2 \le W(t), \quad w(0,t)^2 \le W(t) \tag{26}$$
$$W(t) = M\max_{s\in[-\overline{D},0]}\Big(|X_t(s)|^2 + \|w_t(s)\|^2\Big) \tag{27}$$

for $t \ge \overline{D}$. Therefore, with (23), (25)–(26) and applying Young inequality, one gets for $t \ge \overline{D}$

$$\dot{V}(t) \le -\left(\frac{\underline{\lambda}(Q)}{2} - 4b_2|K|^2\right)|X(t)|^2 - b_1\|w(t)\|^2 - b_2\|v(t)\|^2 - \left(b_1 - 4b_2 - \frac{2|PB|^2}{\underline{\lambda}(Q)}\right)w(0,t)^2 + b_0|\dot{D}(t)| \times \max_{s\in[-\overline{D},0]}\Big(|X_t(s)|^2 + \|w_t(s)\|^2 + \|v_t(s)\|^2\Big)$$

in which $b_0 = b_1(4+2M)+b_2(1+M)$. Consequently, choosing $b_2 = \frac{\underline{\lambda}(Q)}{16|K|^2}$, $b_1 > 4b_2 + \frac{2|PB|^2}{\underline{\lambda}(Q)}$, it follows

$$\dot{V}(t) \leq -\eta V(t) + b|\dot{D}(t)| \max_{s\in[-\underline{D},0]} V(t+s)\,,\ t \geq \overline{D}$$

in which we have introduced $\eta = \frac{\min\left\{\frac{\underline{\lambda}(Q)}{4}, b_2\right\}}{\max\left\{\overline{\lambda}(P), 2b_1\overline{D}\right\}}$ and $b = \frac{b_0}{\min\left\{\underline{\lambda}(P), b_1\underline{D}, b_2(\overline{D}-\overline{D}')\right\}}$. From Lemma 2 in Appendix, one gets the existence of $\delta^* \in (0, \left(\frac{\eta}{b}\right)^2)$ such that, for $\delta < \delta^*$, there exist $r, \gamma > 0$

$$V(t) \leq r \max V_{\overline{D}}\, e^{-\gamma(t-\overline{D})}\,,\ t \geq \overline{D}$$

The fact $\delta^* < 1$ follows conservatively observing that $\eta \leq b_2/(2b_1\overline{D})$ and that $b \geq b_0/(b_1\overline{D}) \geq b_2/(b_1\overline{D})$. Finally, using (22), one deduces (4).

C. Remarks

The main trick enabling to conclude on the overall system convergence without requiring the delay time-derivative to be strictly uniformly bounded by 1 (i.e. $\dot{D}(t) < 1$, $t \geq 0$) is the use of (24) in the analysis (to counteract the appearance of terms $\dot{D}(t)w(0,t)^2$). However, this choice is somewhat conservative in the sense that it only holds for $t \geq \overline{D}$ and hence the exponential result stated in Theorem 1 only holds for $t \geq \overline{D}$.

IV. Simulation results

To illustrate the relevance of the proposed prediction-based control law, we consider the unstable second-order dynamics

$$\dot{X}(t) = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} X(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} U(t - D(t)) \qquad (28)$$

in which $D(t)$ is a communication delay that can be subject to large variations. A schematic view of the system is given in Fig. 1. We consider that the communication between the plant and the controller is not symmetric, resulting only into an input delay[2]. The control law (2) is applied with the feedback gain $K = -[2 \quad 3]$. The integral in (2) is implemented with a trapezoidal discretization scheme.

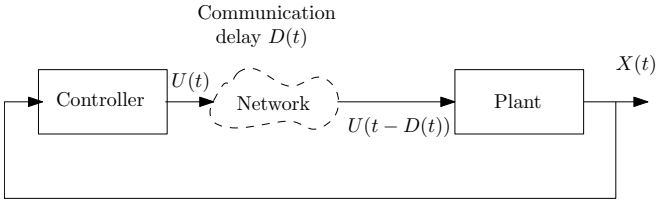

Fig. 1. Schematic view of the considered system with communication delay.

[2]This is usually not the case in practice, as the plant and the controller exchange data through similar channels, which should result into an additional output delay. Note that, for a time-varying delay, an output delay cannot be recast as an input delay. However, our approach can be straightforwardly extended to LTV systems. Therefore, by appropriately modifying the control design, both output- and input-delays could be handled.

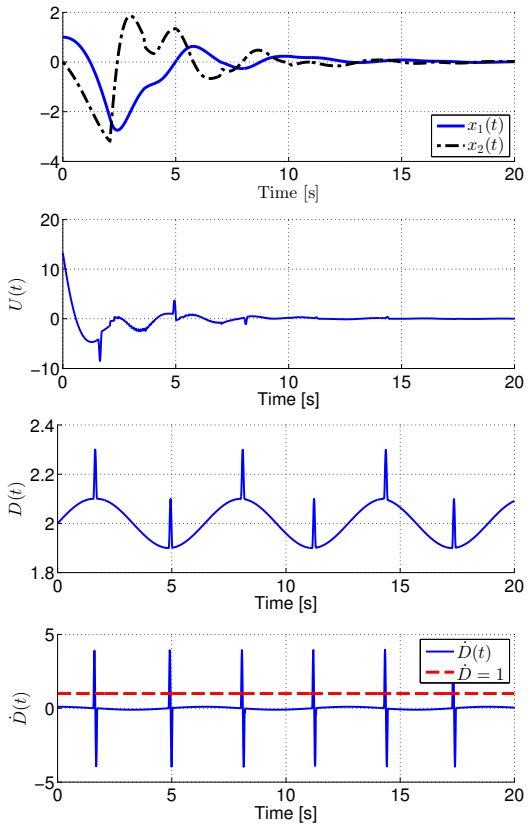

Fig. 2. Closed-loop dynamics for a continuously differentiable delay of sinusoidal form (see the third plot above and Fig. 3).

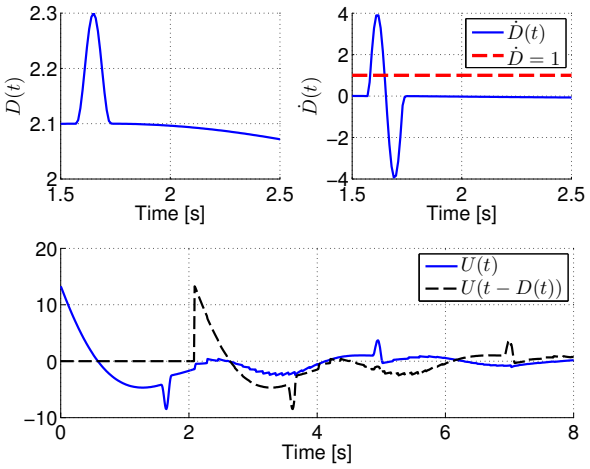

Fig. 3. Zoom-in view of the delay variations and the corresponding delayed control.

We first consider a sinusoidal continuously differentiable delay (see Fig. 3 in which a close-up view of the delay variations is provided), which exhibits time-derivatives greater than one during an infinite number of time intervals Corresponding closed-loop results are pictured in Fig. 2. One can observe that, as Theorem 1 states it can be the case, the plant asymptotically converges despite periodic non-causal delay variations. However, note that, as the delay function is continuously differentiable, a variation of $1-\dot{D}$ from a negative value to a positive one implies that $t - D(t)$ is not injective, i.e. that some inputs are read twice. This has no

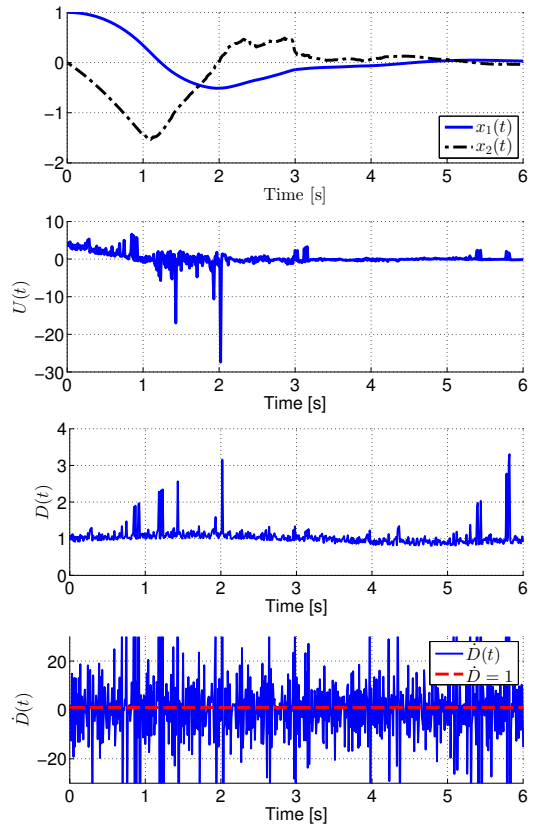

Fig. 4. Closed-loop dynamics with a chattering delay.

physical meaning in the context of a network system.

To evaluate our controller performance in a more meaningful framework, we consider a random delay constrained by the fact that all control inputs are only received once by the plant (i.e. such that there do not exist $t_1 \neq t_2$ s.t. $t_1 - D(t_1) = t_2 - D(t_2)$). Corresponding simulation results are pictured in Fig. 4 and exhibit the same convergence property as previously. In this case, the causality property is almost never satisfied, as can be observed on the fourth plot provided in Fig. 4, but the average delay variations remain sufficiently small for stabilization to be achieved. Despite the fact that the delay is not continuously differentiable, one can observe that the conclusion of Theorem 1 still holds. Hence, extension of our design to almost everywhere (a.e.) piecewise continuously differentiable delay function is a direction of future work.

V. Conclusion

In this work, we presented the first result on prediction-based control for a linear plant subject to a known time-varying input delay which can violate the causality principle. The proposed controller employs a prediction of the system state on a time horizon equal to the current delay value. We have proven that asymptotic convergence is achieved, provided that the average delay variations remain sufficiently small, in the sense of an average $\mathscr{L}_2$-norm.

Our design exploits the fact that the delay is continuously differentiable, which may not be a fairly realistic assumption in the context of communication delays, i.e., for network systems for which each information sent is only received once by the system a priori. This motivates our willingness to extend the design proposed in this paper to piecewise continuously differentiable delay functions.

Appendix

Lemma 2 (Halanay inequality for Time-Varying systems): Consider a positive continuous real-valued function x such that, for some $t_0 \in \mathbb{R}$,

$$\left\{ \begin{array}{rcl} \dot{x}(t) & \leq & -ax(t) + b(t) \max x_t \,, \quad t \geq t_0 \\ x_{t_0} & = & \psi \end{array} \right. \tag{29}$$

with $a \geq 0$, $b : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous function which satisfies

$$\frac{1}{t-h_i} \int_{h_i}^{t} b(s)^2 ds \leq \delta \,, \quad t \in [h_i, h_{i+1}] \tag{30}$$

for some $\delta > 0$ and an ordered sequence $(h_i)_{i\in\mathbb{N}}$ such that $h_0 = t_0$, $\lim_{i\to\infty} h_i = \infty$ and $\Delta h_i = h_{i+1} - h_i \in [\underline{\Delta}, \overline{\Delta}]$ for all $i \in \mathbb{N}$. There exists $\delta^* \in (0, a^2)$ such that, if $\delta < \delta^*$, then there exists $\gamma, r \geq 0$ such that

$$\forall t \geq t_0 \quad x(t) \leq r \max x_{t_0} e^{-\gamma(t-t_0)} \tag{31}$$

Proof: We start our analysis by observing that, without loss of generality, one can consider that $\underline{\Delta} \geq \overline{D} + 1/(2a)$ (otherwise, one can simply consider a subsequence of $(h_i)_{i\in\mathbb{N}}$ which satisfies this property; this one exists as $\lim_{i\to\infty} h_i = \infty$).

Let $t_1 = \inf\left\{ t \geq t_0 \mid x(t) > \max x_{t_0} \right\} \in \mathbb{R} \cup \{\infty\}$ and assume

$$\delta < 4a^2 e^{-1} \triangleq \bar{\delta} < a^2 \tag{32}$$

By definition, $x(t_1) = \max x_{t_0}$, $x(t) < x(t_1)$ for $t < t_1$ and there exists $\varepsilon > 0$ such that $x(t) = \max x_t$ for $t \in [t_1, t_1 + \varepsilon)$. Assume that $t_1 = t_0$. Then (29) rewrites

$$\dot{x}(t) \leq -ax(t) + b(t)x(t) \,, \quad t \in [t_1, t_1 + \varepsilon)$$

and, with (30),

$$\begin{aligned} x(t) &\leq \exp\left(-a(t-t_0) + \int_{t_0}^{t} b(s) ds \right) x(t_0) \\ &\leq \exp\left(-(a - \sqrt{\delta})(t-t_0) \right) \max x_{t_0} \,, \quad t \in [t_1, t_1 + \varepsilon) \end{aligned}$$

From (32), this is in contradiction with the definition of t_1. Hence $t_1 > t_0$. Integrating (29), one gets, for $t \in [t_0, t_1]$,

$$\begin{aligned} x(t) &\leq e^{-a(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{-a(t-s)} b(s) \max x_s ds \\ &\leq \left(e^{-a(t-t_0)} + \int_{t_0}^{t} e^{-a(t-s)} b(s) ds \right) \max x_{t_0} \end{aligned}$$

Applying Cauchy-Schwartz inequality, one concludes that

$$\begin{aligned} x(t) &\leq \left(e^{-a(t-t_0)} + \sqrt{\frac{1 - e^{-2a(t-t_0)}}{2a}} \sqrt{\delta} \sqrt{t - t_0} \right) \max x_{t_0} \\ &\leq \left(e^{-a(t-t_0)} + \sqrt{\frac{\delta}{2a}} \sqrt{t - t_0} \right) \max x_{t_0} \\ &\triangleq \varphi(t - t_0) \max x_{t_0} \,, \; t \in [t_0, t_1] \end{aligned} \tag{33}$$

Studying this function, one can see that φ is strictly increasing on $[0,t_1^*]\cup[t_2^*,\infty)$ and decreasing on $[t_1^*,t_2^*]$ in which $t_1^* < 1/(2a) < t_2^*$ and t_1^*, t_2^* are the two solutions of

$$\sqrt{t}e^{-at} = \frac{1}{2a}\sqrt{\frac{\delta}{2a}} \tag{34}$$

which exist from (32). Further, there exist τ_1^*, τ_2^* such that $t_1^* \le \tau_1^* < 1/2a < t_2^* \le \tau_2^*$ and $\varphi(\tau_1^*) = \varphi(\tau_2^*) = 1$. It follows that $\varphi(t) < 1$ for $t \in (\tau_1^*, t_2^*)$. Besides, one gets that τ_1^* and t_1^* (resp. τ_2^* and t_2^*) are increasing (resp. decreasing) with δ with $\lim_{\delta\to 0}\tau_1^* = \lim_{\delta\to 0} t_1^* = 0$ (resp. $\lim_{\delta\to 0}\tau_2^* = \lim_{\delta\to 0} t_2^* = \infty$).

Now, pick $\bar{\bar{\delta}} > 0$ such that, for $\delta \le \bar{\bar{\delta}}$,

$$\tau_1^* \le \underline{\Delta} \quad \text{and} \quad t_2^* \ge \overline{\Delta} \tag{35}$$

which exists according to the previous considerations, and assume that $\delta < \delta^* \stackrel{\Delta}{=} \min\left\{\bar{\delta}, \bar{\bar{\delta}}\right\}$. Finally, define

$$\underline{\varepsilon} = \underline{\Delta} - \overline{D} - 1/(2a) \text{ and } \varepsilon_0 = \Delta h_0 - \overline{D} - \tau_1^* \tag{36}$$

From (36) and the fact that $\underline{\Delta} > 1/(2a) + \overline{D}$, it follows that $\varepsilon_0 \ge \underline{\varepsilon} > 0$. Consider δ_0 such that $\tau_1^* = \min\{h_1 - t_0, t_1 - t_0\}$ for $\delta = \delta_0$ and assume temporarily that $\delta \le \delta_0$. Then, if $t_1 \le t_0 + t_2^*$, from (33), we conclude that $x(t_1) < \max x_{t_0}$ and we again obtain a contradiction with the definition of t_1. Consequently, $t_1 > t_0 + t_2^* \ge h_1$ from (35). Thus, δ_0 is such that $\tau_1^* = h_1 - t_0$ for $\delta = \delta_0$. From (35), one gets that $\bar{\bar{\delta}} < \delta_0$. Hence, by picking $\delta < \delta^*$, the conclusion $t_1 > h_1$ holds.

Further, from (35), it follows that $\tau_1^* + \varepsilon_0 + \overline{D} < t_2^*$. Hence, to summarize, by construction, one gets (33) and

- $t_1 > h_1$; and
- $\max x_{h_1} \le \varphi(\tau_1^* + \varepsilon_0) \max x_{h_0}$ with $\varphi(\tau_1^* + \varepsilon_0) < 1$.

We now prove similar properties by iterations, for $i \ge 1$. Integrating (29) between h_i and t and following the same lines as previously, one gets

$$x(t) \le \varphi(t - h_i) \max x_{h_i}\,,\ t \in [h_i, h_i + t_2^*]$$

Consider δ_i such that $\tau_1^* = \min\{t_1 - h_i, h_{i+1} - h_i\}$. With the same arguments as those previously used, one obtains that $t_1 > t_2^* + h_i > h_{i+1}$ provided that $\delta \le \delta_i$. Similarly, one can show that this condition holds as $\delta < \delta^* < \delta_i$. Defining $\varepsilon_i = \Delta h_i - \overline{D} - \tau_1^*$, one finally obtains $\max x_{h_{i+1}} \le \varphi(\tau_1^* + \varepsilon_i) \max h_i$ with $\varphi(\tau_1^* + \varepsilon_i) < 1$. More precisely, by direct iterations, one gets, for $t \in [h_i, h_{i+1}]$,

$$x(t) \le \varphi(t - h_i) \prod_{j=1}^{i-1} \varphi(\tau_1^* + \varepsilon_j) \max x_{t_0} \tag{37}$$

It follows directly that $t_1 = \infty$. Further, from (37) and as $\varepsilon_i > \underline{\varepsilon} > 0$ for all $i \in \mathbb{N}$, there exist $r, \gamma > 0$ such that (31) is satisfied. ■

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