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# An iterative algorithm for dynamic optimization of systems with input-dependent hydraulic delays

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Abstract
In this article, we propose a numerical algorithm capable of handling optimal control problems for a class of systems with input-dependent input hydraulic delays. Such delays are often observed in process industries. A careful look at the stationarity conditions allows us to derive an iterative algorithm approaching the solution of this problem by solving a series of simpler auxiliary instances. Interestingly, the algorithm is able to leverage state-of-the-art numerical optimization tools such as IPOPT. The proof of convergence is sketched, highlighting the relevance of the chosen algorithmic structure as a form of gradient descent in a functional space. The practical interest of the algorithm is evidenced on a numerical example, showing the desirable properties of convergence and the numerical efficiency.

*Keywords:* Optimal control, time varying delays, numerical methods, convergence analysis.

## 1. INTRODUCTION

Many applications in the field of process control use dynamic optimization in order to treat systems with time delays (*cf* Richard (2003)). For this reason, most commercial Model Predictive Control (MPC) tools routinely take into account fixed time delays, and implementations are common place in industrial applications. Practically, delays are usually treated directly in the time-discretization schemes. Formally, the optimality conditions have been investigated early on by the control system community, see Halanay (1968), Soliman and Ray (1971), Malez-Zavarei (1980), Basin and Rodriguez-Gonzales (2006), Frederico and Torres (2012). A detailed panorama of the available stationarity conditions (including the case of state-constrained problems) can be found in Gollmann et al. (2009) and Göllmann and Maurer (2014) which also propose numerical methods for implementation. These works cover cases of multiple input and state delays in Pontryagin's maximum principle. Besides MPC techniques, other approaches have focused on optimal synthesis, for fixed delays, resulting in feedback control laws. Many research efforts have focused on related numerical aspects, *e.g.* investigations regarding the stability of some orthogonal collocation schemes with regard to the transcription of systems of delay algebraic equations have also been investigated (see Betts et al. (2015)).

Interestingly enough, it appears that fewer attention has been given to dynamic optimization problems under varying delays. This topic is not new however and since the seminal work of Banks (1968), most research efforts have focused on closed-form solutions to LQR problems for dynamics impacted by time-varying delays, see Carravetta et al. (2010). However, these approaches usually do not take into account cases where the delay variability actually depends on the input or the state. Furthermore, in most practical applications where delays are *a priori* known to be variable, this information is simply ignored and the delays are assumed to be fixed. Problems where such delay dependency in the control is important are nevertheless numerous and of great practical importance as the case of input-dependant hydraulic delays frequently arises in the plug-flow modelling of fluid transport phenomena. Examples of such systems in the process industries can be found in Harmand and Dochain (2005), Depcik and Assanis (2005), Chèbre and Pitollat (2008), Roca et al. (2008), Zenger and Niemi (2009), Bresch-Pietri et al. (2014), Petit (2015) or M. Sbarciog and Prada (2008). The first theoretical results regarding the stationarity conditions of this type of systems were laid out in Clerget et al. (2016). However, as evidenced in Clerget et al. (2017), a straightforward direct simultaneous approach using orthogonal

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collocations (*e.g.* Biegler (2007), A. Flores-Tlacuahuac and Biegler (2008) or Biegler (2010)) fails on this type of applications. Indeed, the dependency of the delay with respect to the input does not allow to transcript the continuous optimal control problem as a smooth NLP [1]. To circumvent this issue, Clerget et al. (2017) attempted to discretize spatially the underlying plug-flow advection partial differential equation (PDE) to proceed with the direct optimization of the subsequent finite dimension model, as has been studied for the modelling of more complex transport systems *e.g.* Agarwal (2010). However, in their numerical findings, the authors emphasize the sensitivity of the results with respect to the choice of the descretization scheme of the PDE. Overall, the numerical performances are not fully satisfactory as a good numerical accuracy of the PDE's discretization must be paid by an increased state size leading to a large computational load and high index algebraic equations if state constraints are to be imposed. Typical resolution times range from tens of seconds to a couple of minutes. The method is also shown to be prone to numerical difficulties as a refined spatial discretization of the PDE leads to the ill-conditioning of the Lagrangian's Hessian that the solver must invert. This leads to a malicious game where the numerical performances of the solver deteriorate as the dimension of the problem, and its inherent difficulty, increases.

In this paper, we propose an alternative methodology to achieve the optimal control of systems displaying input-dependant input delays based on an iterative procedure. We will begin by introducing the stationarity conditions of the optimal control problem that we consider. We will then use those conditions to derive a candidate iterative algorithm to solve the optimal control problem. We will then sketch its proof of convergence by showing that it can be viewed, in a limit case, as a gradient descent algorithm. Finally, we will present and discuss numerical results, based on a simple benchmark problem from Bresch-Pietri et al. (2014), illustrating the performances of the algorithm.

## 2. NOTATIONS AND PROBLEM STATEMENT

Let $T > 0$ and $n \in \mathbb{N}^*$, we note $L^2([0;T], \mathbb{R}^n)$ the space of functions of integrable square over the interval $[0;T]$ and $D^1([0;T], \mathbb{R}^n)$ the space of continuous and differentiable functions on the interval $[0;T]$.

Let $x_0 \in \mathbb{R}^m$ and $P \in \mathcal{M}_p(\mathbb{R})$ be symmetric definite positive. Let $\phi : \mathbb{R}^p \to \mathbb{R}_+^*$, $L : [0;T] \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ and $f : [0;T] \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^m$ be smooth functions. Take $(v_0, u_0) \in L^2([r_0;0], \mathbb{R}^p) \times D^1([r_0;0], \mathbb{R}^p)$, $r_0 < 0$ with

$$\int_{r_0}^{0} \phi\left(u_0(\tau)\right) \mathrm{d}\tau = 1$$

and

$$\forall t \in [r_0;0],\ u_0(t) = u_0(0) + \int_0^t v_0(\tau)\,\mathrm{d}\tau$$

**Problem statement :** Our goal in this paper is to solve the following optimal control problem (generalization of a Bolza problem)

$$\begin{aligned} \mathcal{P} : \min_v \int_0^T & L(t, x(t), u(t)) + \frac{1}{2} v(t)^T P v(t)\,\mathrm{d}t \triangleq J(v) \\ s.t.\quad & \forall t \in [0;T],\ \dot{x}(t) = f(t, x(t), u(t), u(r_u(t))) \\ & \forall t \in [0;T],\ \dot{u}(t) = v(t) \\ & x(0) = x_0,\ u_{[r_0;0]} = u_0,\ v_{[r_0;0]} = v_0 \end{aligned}$$

where the hydraulic delay $D_u(t) \triangleq t - r_u(t)$ is implicitly defined by the relation

$$\int_{r_u(t)}^{t} \phi(u(\tau))\,\mathrm{d}\tau = 1 \tag{1}$$

and in particular

$$r_0 = r_u(0)$$

Equation (1) defines an hydraulic delay impacting the smooth variable $u$ which is the input of the system having $x$ as state.

In the following, in addition with the classic notations $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial u}$, we will denote $\frac{\partial f}{\partial u_r}$ the vector of partial derivatives of $f$ w.r.t. its last argument.

## 3. ALGORITHM DESIGN

Let us consider the operator $\mathfrak{P} : L^2([0;T], \mathbb{R}^p) \to D^1([0;T], \mathbb{R}^p) \times D^1([0;T], \mathbb{R}^m)^2 \times D^1([0;T], \mathbb{R}^p)$ where $\mathfrak{P}(v) = (u, x, \lambda, \nu)$ is defined by

$$\dot{u}(t) = v(t),\ u_{[r_0;0]} = u_0 \tag{2}$$

$$\dot{x}(t) = f(t, x(t), u(t), u(r_u(t))),\ x(0) = x_0 \tag{3}$$

$$\begin{aligned} \dot{\lambda}(t) = & -\frac{\partial L}{\partial x}(t, x(t), u(t))^T \\ & - \frac{\partial f}{\partial x}(t, x(t), u_n(t), u(r_u(t)))^T \lambda(t) \end{aligned} \tag{4}$$

$$\lambda(T) = 0$$

$$\begin{aligned} \dot{\nu}(t) = & -\frac{\partial L}{\partial u}(t, x(t), u(t))^T \\ & - \frac{\partial f}{\partial u}(t, x(t), u(t), u(r_u(t)))^T \cdot \lambda(t) \\ & - \mathbb{1}_{[0;r_u(T)]}(t)(r_u^{-1})'(t) \cdot \\ & \quad \frac{\partial f}{\partial u_r}(r_u^{-1}(t), x(r_u^{-1}(t)), u(r_u^{-1}(t)), u(t))^T \cdot \\ & \quad \lambda(r_u^{-1}(t)) \\ & - \int_t^{r_u^{-1}(\min(t, r_u(T)))} \lambda(\tau)^T \cdot \\ & \quad \frac{\partial f}{\partial u_r}(\tau, x(\tau), u(\tau), u(r_u(\tau))) \\ & \quad \frac{v(r_u(\tau))}{\phi(u(r_u(\tau)))}\,\mathrm{d}\tau \frac{\partial \phi}{\partial u}(u(t)) \end{aligned} \tag{5}$$

$$\nu(T) = 0$$

Using these notations [2], it is straightforward to extend the results of Clerget et al. (2016) to show that the stationarity conditions of $\mathcal{P}$ are given by

$$\begin{aligned} (u, x, \lambda, \nu) &= \mathfrak{P}(v) \\ Pv + \nu &= 0 \end{aligned} \tag{6}$$

As mentioned previously, the input-dependency of the delay makes $\mathcal{P}$ impractical to solve directly using orthogonal

[1] In the sense that this dependency leads to index commutations in the equations of the discretized dynamics

[2] On the interval $[0; r_u(T)]$ covered by the indicator function, the function $r_u^{-1}$ employed in (5) is well defined

collocations. Instead, we would much prefer to solve a sequence of simpler auxiliary problems and a natural idea would be to define a sequence of problems displaying a delay law which would be time-varying, but in a fixed fashion based on the value of $v_n$ found at the previous iteration

$$\begin{aligned}\min_{v_{n+1}} \int_0^T & L(t, X_{n+1}(t), u_{n+1}(t)) + \frac{1}{2} v_{n+1}(t)^T P v_{n+1}(t)\, \mathrm{d}t \\ s.t.\quad & \dot{X}_{n+1}(t) = f(t, X_{n+1}(t), u_{n+1}(t), u_{n+1}(r_{u_n}(t))) \\ & \dot{u}_{n+1}(t) = v_{n+1}(t) \\ & x(0) = x_0,\ u_{n+1[r_0;0]} = u_0,\ v_{n+1[r_0;0]} = v_0\end{aligned}$$

Mathematically, the stationarity conditions of each of these problems could then be expressed as special cases of (2)-(5) in which $\phi$ would no longer be a function of $u$ but instead of $t$ alone. However, examining equation (5) shows that the distributed term that it involves (the last term of (5)) would vanish (exactly, at each step) and that if the sequence was ever to converge, its solution would not verify the original stationarity conditions of $\mathcal{P}$, but a biased version of them. Following this remark for all $n \geq 1$, we define

$$\begin{aligned}\mathcal{P}_{n+1} : \min_{v_{n+1}} \int_0^T & L(t, X_{n+1}(t), u_{n+1}(t)) \\ & + \frac{1}{2} v_{n+1}(t)^T P v_{n+1}(t) \\ & + \mathcal{S}_n(t)(u_{n+1}(t) - u_n(t)) \\ & + \frac{\alpha}{2} \| v_{n+1}(t) - v_n(t) \|_2^2\, \mathrm{d}t \\ s.t.\quad & \dot{X}_{n+1} = \\ & \quad f(t, X_{n+1}(t), u_{n+1}(t), u_{n+1}(r_{u_n}(t))) \\ & \dot{u}_{n+1} = v_{n+1} \\ & X_{n+1}(0) = x_0,\ u_{n+1[r_0;0]} = u_0, \\ & v_{n+1[r_u(0);0]} = v_0\end{aligned}$$

where

$$\begin{aligned}\mathcal{S}_n(t) = \int_t^{r_{u_n}^{-1}(\min(t, r_{u_n}(T)))} & \lambda_n(\tau)^T \cdot \\ & \frac{\partial f}{\partial u_r}(\tau, x_n(\tau), u_n(\tau), u_n(r_{u_n}(\tau))) \cdot \\ & \frac{v_n(r_{u_n}(\tau))}{\phi(u_n(r_{u_n}(\tau)))}\, \mathrm{d}\tau \frac{\partial \phi}{\partial u}(u_n(t))\end{aligned}$$

is the sensitivity of the objective with respect to the change of the delay law caused by a change of the control input as derived from the calculus of variations. In the definition of $\mathcal{S}_n$ and the general statement of $\mathcal{P}_n$, $(u_n, x_n, \lambda_n, \nu_n)$ are defined as

$$v_n \mapsto (u_n, x_n, \lambda_n, \nu_n) \triangleq \mathfrak{P}(v_n)$$

Throughout the rest of the discussion, the following assumptions are considered

*Assumption 1.* $L$ is twice continuously differentiable while $f$, $\phi$ are continuously differentiable. There exists $K \geq 0$ such that

$$\forall (t, x, u) \in [0; T] \times \mathbb{R}^m \times \mathbb{R}^p,\ \|\nabla^2 L(t, x, u)\|_1 \leq K$$

and

$$\forall (t, x, u, u_r) \in [0; T] \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^p,\ \|\nabla f(t, x, u, u_r)\|_1 \leq K$$

and

$$\forall u \in \mathbb{R}^p,\ \|\nabla \phi(u)\|_1 \leq K$$

and, $\nabla^2 L$, $\nabla f$, $\nabla \phi$ are K-Lipschitz continuous.

*Assumption 2.* There exists $J^* \in \mathbb{R}$ such that

$$\forall v \in L^2([0; T]),\ J^* \leq J(v)$$

*Assumption 3.* There exists $\phi_{\min} > 0$ such that

$$\forall u \in \mathbb{R},\ \phi_{\min} \leq \phi(u)$$

*Remark 4.* Assumptions 1-2 are classic in optimization. Assumption 3 is usually considered for systems with input varying delays of hydraulic type (see Bresch-Pietri et al. (2014)) so that $r'_u$ be bounded away from zero and the input keep on reaching the plant.

*Definition 5.* Consider a sequence $(v_n)_{n \in \mathbb{N}^*}$ and $\alpha \geq 0$, $(v_n)$ is called $\alpha$-admissible if for all $n \geq 2$, $v_n$ is a solution (possibly local) of $\mathcal{P}_n$.

Let us define

$$\begin{aligned}\mathcal{X} \triangleq \{ v \in L^2([0; T]),\ \exists R_v \in \mathbb{R}_+,\ \forall w \in L^2([0; T]), \\ J(w) \leq J(v) \implies \|w\|_2 \leq R_v \}\end{aligned} \quad (7)$$

the set of $L^2$ functions such that their $J$-level set is included in a ball of $L^2$ and note

$$g_v \triangleq P v + \nu \quad (8)$$

The main result concerning the sequence $(\mathcal{P}_n)$ is as follows

*Theorem 6.* Under Assumptions 1, 2 and 3, given any $\alpha$-admissible sequence $(v_n)_{n \in \mathbb{N}^*}$ such that $v_1 \in \mathcal{X}$, if $\alpha$ is large enough then $(v_n)$ satisfies

$$\lim_{n \to \infty} \|g_{v_n}\|_2 = 0$$

and

$$\lim_{n \to \infty} \|v_{n+1} - v_n\|_2 = 0$$

Furthermore, the sequence $(J(v_n))_{n \in \mathbb{N}^*}$ is monotonically decreasing.

**Proof.** Given $n \in \mathbb{N}^*$, let us assume that $v_n \in \mathcal{X}$ (which is true for $n = 1$ by assumption) and, by extension of (7), define

$$\mathcal{X}_n \triangleq \{ v \in L^2([0; T]),\ J(v) \leq J(v_n) \} \subset \mathcal{X}$$

which is a bounded set in the sense of the $L^2$ norm, *i.e.* there exists $R_n > 0$ such that

$$\forall v \in \mathcal{X}_n,\ \|v\|_2 \leq R_n \quad (9)$$

Consider the operator $\mathfrak{Q} : L^2([0; T], \mathbb{R}^p)^2 \to D^1([0; T], \mathbb{R}^p)^2 \times D^1([0; T], \mathbb{R}^m)^2$ with $\mathfrak{Q}(v, w) = (u, q, x, \lambda)$ defined as

$$\dot{u}(t) = v(t),\ u_{[r_0;0]} = u_0 \quad (10)$$

$$\dot{q}(t) = w(t),\ q_{[r_0;0]} = u_0 \quad (11)$$

$$\dot{x}(t) = f(t, x(t), u(t), u(r_q(t))),\ x(0) = x_0 \quad (12)$$

$$\begin{aligned}\dot{\lambda}(t) = & -\frac{\partial L}{\partial x}(t, x(t), u(t))^T \\ & - \frac{\partial f}{\partial x}(t, x(t), u(t), u(r_q(t)))^T \lambda(t)\end{aligned} \quad (13)$$

$$\lambda(T) = 0$$

Note the slight (but important) differences between $\mathfrak{P}$ defined by (2)-(5) and $\mathfrak{Q}$. The second argument of $\mathfrak{Q}$ is used to define the time-varying delay appearing in the right-hand side of equations (12)-(13).

The newly defined operator $\mathfrak{Q}$ plays a key role with respect to the sequence $(v_n)$. Indeed, the stationarity conditions of $\mathcal{P}_{n+1}$ are given by

$$
\begin{aligned}
&(u_{n+1}, X_{n+1}, \Lambda_{n+1}) = \mathfrak{Q}(v_{n+1}, v_n)\\
&\dot{N}_{n+1}(t) =\\
&\quad -\frac{\partial L}{\partial u}(t, X_{n+1}(t), u_{n+1}(t))^T\\
&\quad -\frac{\partial f}{\partial u}(t, X_{n+1}(t), u_{n+1}(t), u_{n+1}(r_{u_n}(t)))^T \Lambda_{n+1}(t)\\
&\quad -\mathbb{1}_{[0;r_{u_n}(T)]}(t)(r_{u_n}^{-1})'(t)\cdot\\
&\quad\quad \frac{\partial f}{\partial u_r}(r_{u_n}^{-1}(t), X_{n+1}(r_{u_n}^{-1}(t)), u_{n+1}(r_{u_n}^{-1}(t)), u_{n+1}(t))^T\cdot\\
&\quad\quad \Lambda_{n+1}(r_{u_n}^{-1}(t)) - \mathcal{S}_n(t)^T\\
&N_{n+1}(T) = 0\\
&0 = Pv_{n+1} + N_{n+1} + \alpha(v_{n+1} - v_n)
\end{aligned}
\tag{14}
$$

Noticing that

$$(\mathfrak{Q}(v, v), N(\mathfrak{Q}(v, v), v)) = \mathfrak{P}(v)$$

it is clear from the structure of $\mathcal{P}_{n+1}$ that if an $\alpha$-admissible sequence $(v_n)$ converges, its limit will satisfy equations (2)-(5).

From (14), we directly deduce that the solutions of $\mathcal{P}_n$ and $\mathcal{P}_{n+1}$ are related by

$$v_{n+1} = v_n - \frac{1}{\alpha} g_{v_n} + \frac{1}{\alpha}\epsilon_{n+1} \tag{15}$$

with

$$\epsilon_{n+1} = -P(v_{n+1} - v_n) - (N_{n+1} - \nu_n)$$

In turn, the cost variation between $v_n$ and $v_{n+1}$ is given by

$$J(v_{n+1}) - J(v_n) = \int_0^1 G'(s)\,\mathrm{d}s$$

where

$$G(s) = J(v_n + (v_{n+1} - v_n)s)$$

Using the adjoint state method (see *e.g.* Strang (2007)), one computes, after a few lines of calculus,

$$
\begin{aligned}
&J(v_{n+1}) - J(v_n) =\\
&\quad \int_0^1 \int_0^T g_{v_n + (v_{n+1} - v_n)s}(t)^T (v_{n+1}(t) - v_n(t))\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}
$$

which gives

$$
\begin{aligned}
&J(v_{n+1}) - J(v_n) =\\
&\quad -\frac{1}{\alpha}\|g_{v_n}\|_2^2 + \frac{1}{\alpha}\langle g_{v_n}, \epsilon_{n+1}\rangle\\
&\quad + \int_0^1 \langle g_{v_n + (v_{n+1} - v_n)s} - g_{v_n}, v_{n+1} - v_n\rangle\,\mathrm{d}s
\end{aligned}
$$

Finally

$$
\begin{aligned}
&J(v_{n+1}) - J(v_n) \leq\\
&\quad -\frac{1}{\alpha}\|g_{v_n}\|_2^2 + \frac{1}{\alpha}\|g_{v_n}\|_2\|\epsilon_{n+1}\|_2\\
&\quad + \int_0^1 \|g_{v_n + (v_{n+1} - v_n)s} - g_{v_n}\|_2 \|v_{n+1} - v_n\|_2\,\mathrm{d}s
\end{aligned}
\tag{16}
$$

At this point, the two main technical results remaining to establish the desired results are to show that $\|\epsilon_{n+1}\|_2$ admits an upper bound proportional to $\|g_{v_n}\|_2$ and that the function $v \mapsto g_v$ is Lipschitz continuous with respect to the $L^2$ norm on any bounded set. For the sake of briefness, the explicit derivation of these properties is left to a forthcoming publication.

Then, one shows that for $\alpha$ large enough, $J(v_{n+1}) - J(v_n) < 0$. In particular, this guarantees that $v_{n+1} \in \mathcal{X}_n$. By induction, this implies that if one picks a value $\alpha = \alpha_1$ such that it guarantees a decrease in cost at $n = 1$, then for all rank $n$, $v_n \in \mathcal{X}_1$ and there exists $\beta > 0$ such that

$$\forall n \in \mathbb{N}^*,\ J(v_{n+1}) - J(v_n) \leq -\beta\|g_{v_n}\|_2^2$$

This leads to

$$\sum_{i=0}^{N} \|g_{v_i}\|_2^2 \leq \beta(J(v_0) - J(v_{n+1}))$$

Finally we derive

$$\sum_{i=0}^{N} \|g_{v_i}\|_2^2 \leq \beta(J(v_0) - J^*)$$

and the convergence of the series yields

$$\lim_{n\to\infty} \|g_{v_n}\|_2 = 0$$

which gives the conclusion.

## 4. NUMERICAL EXAMPLE

For the sake of illustration, we treat a benchmark problem already considered in Bresch-Pietri et al. (2014). Consider a second order unstable system with dynamics given by

$$
\begin{aligned}
&\ddot{x}(t) - \dot{x}(t) + x(t) = u(r_u(t))\\
&\dot{u}(t) = v(t)
\end{aligned}
\tag{17}
$$

having the following initial conditions

$$
\begin{aligned}
&x(0) = 1,\ \dot{x}(0) = 0\\
&u_{[r_0;0]} = 1,\ v_{[r_0;0]} = 0
\end{aligned}
$$

This can equivalently be recast as

$$
\begin{aligned}
&\dot{X}(t) = AX(t) + Bu(r_u(t))\\
&\dot{u}(t) = v(t)
\end{aligned}
$$

where

$$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The optimal control problem is

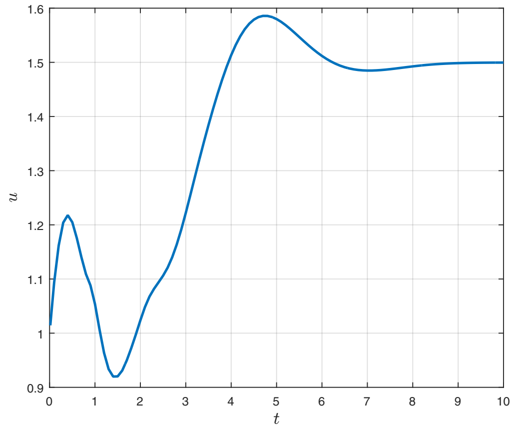
$$
\begin{aligned}
\mathcal{P} : \min_v \int_0^T &\|x(t) - x_r\|_2^2 + w_u\|u(t) - u_r\|_2^2 + w_v\|v(t)\|_2^2\,\mathrm{d}t\\
s.t.\ \ &\dot{X}(t) = AX(t) + Bu(r_u(t))\\
&\dot{u}(t) = v(t)
\end{aligned}
$$

with $T = 10$, $w_u = 0.1$, $w_v = 0.1$ and $x_r = u_r = 1.5$. Given $\alpha = 5$, we iteratively approach a solution of $\mathcal{P}$ by constructing an $\alpha$-admissible sequence $(v_n)$. We pick the trivial initialization value $v_1 = 0$ and for all $n \geq 1$ apply the following algorithm :

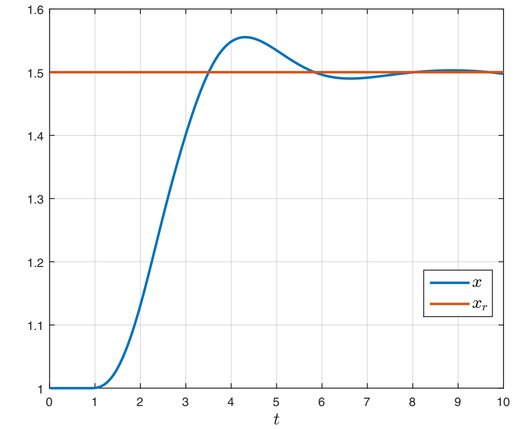
- given $v_n$, compute $u_n$ and the delay law $r_n$
- compute $(x_n, \lambda_n)$ and deduce $\mathcal{S}_n$
- solve $\mathcal{P}_{n+1}$ and obtain $v_{n+1}$

Practically, the resolution of $\mathcal{P}_{n+1}$ is performed using a direct collocation transcription method, AMPL as algebraic modelling language and IPOPT 3.11.8 as NLP solver. The time horizon is divided into 100 finite elements of equal size, each of them containing 3 Radau collocation points. The results are presented on Figures 1-4.

Figures 1-2 display the optimal trajectory that is computed and the associated delay law (the values are obtained for $n = 100$). The inflexions of the input profile




Input




Output

Figure 1. Optimal trajectory computed for $\mathcal{P}$

around $t = 0.9$ and $t = 2.4$ are typical of systems having variable delays. Figure 3 pictorially shows how this trajectory is progressively approached by the sequence. Figure 4 exhibits some indicators regarding the convergence properties of the algorithm as $n$ grows : the cost $J$ along with the relative steps size measured by $\log_{10}(\Delta v) \triangleq \log_{10}(\frac{\|v_n - v_{n-1}\|_2}{\|v_n\|_2})$ and $\log_{10}(\Delta J) \triangleq \log_{10}(\frac{\|J_n - J_{n-1}\|_2}{\|J_n\|_2})$ at successive iterations. As expected, the cost decreases monotonically and the linear shape of the cost decrease on the semi-log plot is evocative of a first order steepest descent-like method. The total computation time for the first 100 iterations displayed on Figure 4 using a 2.60 GHz Intel(R) Core(TM) i7-4720HQ processor on a 64 bits system with a 16.0 Go RAM is equal to 12.61 seconds, 9.78 seconds being actually spent in the solver.

## 5. CONCLUSION

In this paper, we have proposed an iterative algorithm to solve the problem of optimal control of systems with hydraulic input-dependent input delays. A convergence proof was sketched and numerical results were given illustrating the practical interest of the method. More details will be given in a forthcoming publication.

A straightforward extension would be to extend the calculus of variations and the deduced iterative optimization

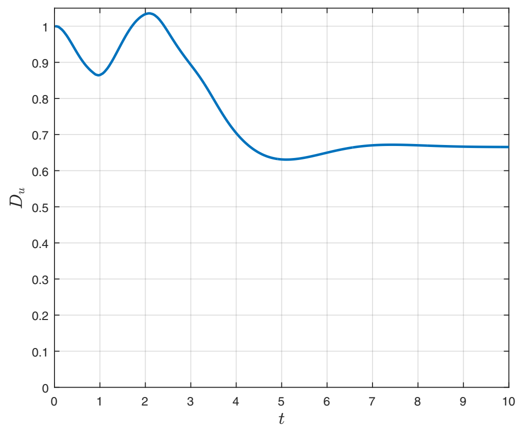
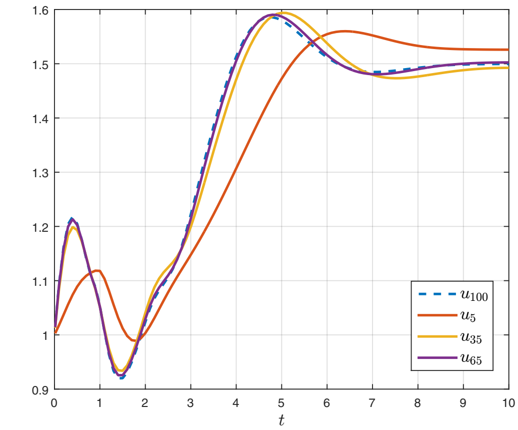
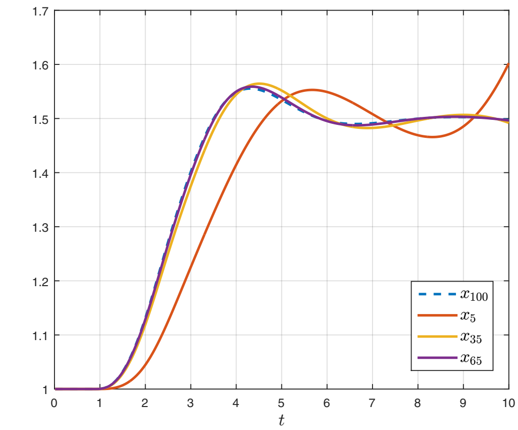



Figure 2. Delay law of the optimal trajectory




Input convergence

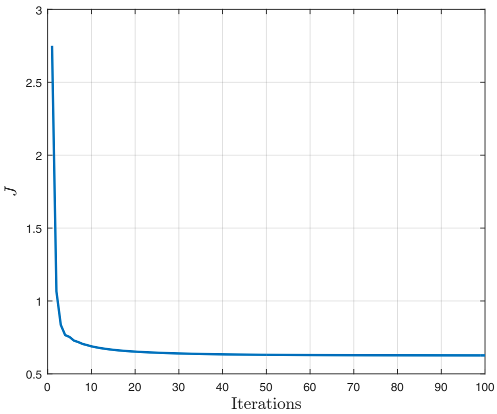



Output convergence

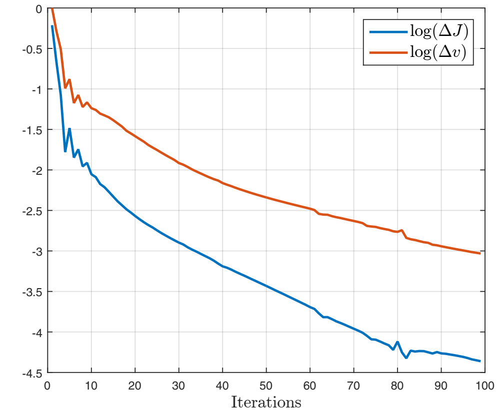
Figure 3. Successive approximations of the optimal trajectory

algorithm to the case of systems with hydraulic input-dependent state delays. This case is of importance since it is instrumental in the modelling of recycling loops or cascades of reacting units.

The current study has focused on the open-loop generation of optimal trajectories for the system. A valuable improvement would be to study the closed-loop behaviour of such a methodology used in a receding horizon framework for




Cost decrease




Size of the steps

Figure 4. Convergence properties of the algorithm

real-time control applications. Stability conditions for such an MPC scheme could be fruitfully investigated.

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