On Input-to-State Stability of Linear Difference Equations and Its Characterization with a Lyapunov Functional

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Abstract: This paper investigates necessary and sufficient Lyapunov conditions for Input-to-State Stability (ISS) of Linear Difference Equations with pointwise delays and an additive exogenous signal. Grounding on recent works in the literature on necessary conditions for the exponential stability of such Difference Equations, we propose a quadratic Lyapunov functional involving the derivative of the so-called delay Lyapunov matrix of the corresponding homogeneous Difference Equation. We prove that the ISS of Linear Difference Equations is equivalent to the existence of an ISS Lyapunov functional. We apply this result to the stability and ISS analysis of hyperbolic Partial Differential Equations of conservation laws.

Keywords: Delay systems, Difference Equations, Input-to-State Stability, Lyapunov Theorem, Distributed Parameter Systems

1. INTRODUCTION

Difference Equations constitute a class of delay systems that has been seldom studied in the literature. Nevertheless, it has been long noticed, with the earliest link going back to d'Alembert formula, that they can be used to represent a broad class of hyperbolic Partial Difference Equations (PDEs). These include conservation laws (see (Bastin and Coron, 2016)) and wave equations used to model transport and propagation phenomena, such as thermal exchanges occurring, for instance, for automotive engines or acoustic systems, to name a few. Recently, the exact relation between Linear First-Order Hyperbolic PDEs and Linear Difference Equations (LDEs) has been comprehensively studied in (Auriol and Di Meglio, 2019).

In this paper, we are interested in the Input-to-State Stability (ISS) of LDEs with pointwise delays with respect to an additive exogenous signal. We wish to study if one can characterize this property with a Lyapunov functional, as it is possible for neutral functional differential equations (see (Pepe and Karafyllis, 2013; Pepe, 2014)), for instance. This question arose in our recent work (see (Auriol and Bresch-Pietri, 2022)) investigating robust feedback for an underactuated network of interconnected PDEs actuated at the boundary, as the one appearing in mining ventilation systems such as in (Rodriguez-Diaz et al., 2021) or oil production systems consisting of networks of pipes. Indeed, to study the cascade of these PDEs, one wishes to consider the equivalent LDE and rely on Lyapunov ISS functionals to investigate the effect of the cascade, as commonly done in a small-gain context, for instance.

Nevertheless, while the Input-to-State Stability of a large number of PDEs with bounded control operator or admissible boundary control is now well-grounded (see (Mironchenko and Prieur, 2020) for a complete review of this field) and its characterization with a coercive ISS Lyapunov function clearly investigated, it is not the case for Difference Equations. Indeed, to our best knowledge, some of the only works investigating this question are (Hale and Verduyn Lunel, 1993) and (Karafyllis and Krstic, 2014). On the one hand, (Hale and Verduyn Lunel, 1993) proved that the asymptotic stability of the homogeneous LDE is equivalent to the ISS of the non-homogeneous one (via Duhamel's principle) but did not consider Lyapunov characterization. On the other hand, (Karafyllis and Krstic, 2014) proposed Lyapunov ISS conditions of general nonlinear Difference Equations, but which are only sufficient. ISS Lyapunov characterizations for nonlinear continuous-time difference equations are provided in (Pepe, 2014) in terms of Lyapunov functional continuous-time difference operator

In this paper, necessary and sufficient Lyapunov conditions for ISS of LDEs with pointwise delays are thus investigated. Grounding on the recent work of (Rocha Campos et al., 2018) on necessary Lyapunov conditions for the exponential stability of such LDEs, we propose a quadratic functional involving the derivative of the so-called delay Lyapunov matrix of the homogeneous system. With a careful analysis of the discontinuities of this matrix and adequate modifications of the Lyapunov functional constructed in (Rocha Campos et al., 2018), we prove that the ISS of a LDE is equivalent to the existence of an ISS

Lyapunov functional. This is the main contribution of the paper.

The paper is organized as follows. Section 2 first presents the problem under consideration, along with preliminary properties. Then, a Lyapunov-Krasovskii functional is constructed in Section 3, in which the main theorem of the paper, that is, the equivalence between ISS of a LDE and existence of an ISS Lyapunov functional, is stated. Finally, Section 4 focuses on applying this result to the stability and ISS analysis of hyperbolic PDEs of conservation laws before drawing perspectives of future works in Section 5.

Notations: A function f is said to be piecewise continuous (resp. constant) on an interval $[a,b] \subset \mathbb{R}$ if the interval can be partitioned by a finite number of points $(t_i)_{0\leq i\leq n}$ so that f is continuous (resp. constant) on each subinterval (t_{i-1},t_i) and admits finite right-hand and left-hand limits at each t_i. A function f is said to be piecewise constant on $\mathbb{R}$ or $\mathbb{R}_+$ if its restriction to any interval is piecewise constant. For any fixed $\tau > 0$, we denote $C_\tau^{pw} = C^{pw}([-\tau,0),\mathbb{R}^n)$ the Banach space of piecewise continuous functions mapping the interval $[-\tau,0)$ into $\mathbb{R}^n$. For a function $\varphi : [-\tau,\infty) \mapsto \mathbb{R}^n$, we define its partial trajectory $\varphi_{[t]}$ by $\varphi_{[t]}(\theta) = \varphi(t+\theta), -\tau \leq \theta \leq 0$. The space C_τ^{pw} is endowed with the norm $||\varphi||^2_{C_\tau^{pw}} = \sup_{s\in[-\tau,0)} \varphi^T(s)\varphi(s)$ or with the L^2_τ norm $||\varphi||^2_{L^2_\tau} = \int_{-\tau}^0 \varphi^T(s)\varphi(s)ds$. The set of natural numbers is denoted by $\mathbb{N}$. For all positive integers p and q, we denote $\mathfrak{M}_{p\times q}(\mathbb{R})$ the set of real matrices with p rows and q columns. The identity matrix of size $n \in \mathbb{N}$ is denoted Id_n or Id, when no confusion arises. We denote $||\cdot||$ the usual Euclidean norm. For real matrices, the induced norm is used. The Dini upper right-hand derivative of a functional $v(\varphi_{[t]})$ is denoted by $D^+v(\varphi_{[t]})$.

2. PROBLEM UNDER CONSIDERATION AND PRELIMINARY RESULTS

2.1 Presentation of the system

Consider $M \in \mathbb{N}\backslash\{0\}$ and positive time-delays $\tau_k > 0$ $(1 \leq k \leq M)$ ordered as $0 < \tau_1 < \tau_2 < ... < \tau_M$. Introduce the following non-homogeneous difference system

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k) + f(t), \quad t \geq 0 \tag{1}$$

where $A_k \in \mathfrak{M}_{n\times n}(\mathbb{R})$. The initial data are given by $X^0 \in C^{pw}_{\tau_M}$. The function f is an exogenous signal which belongs to $C^{pw}([0,\infty),\mathbb{R}^n)$. A function $X : [-\tau_M,\infty) \to \mathbb{R}^n$ is called a solution of the initial value problem (1) if $X_{[0]} = X^0$ and if equation (1) is satisfied for $t \geq 0$. The solution at time t of the system (1) with initial condition X^0 is denoted by $X(t,X^0)$. The dependence with respect to X^0 may be dropped when no confusion arises. We define the homogenous system associated to (1) as

$$X(t) = \sum_{k=1}^{M} A_k X(t-\tau_k), \tag{2}$$

with the initial data $X^0 \in C^{pw}_{\tau_M}$. In this paper, we use the following definition of stability

Definition 1. System (2) is said to be (L^2-)*exponentially stable* if there exist $\mu > 0$ and $\beta \geq 0$ such that for any $X^0 \in C^{pw}_{\tau_M}$, we have

$$||X_{[t]}||_{L^2_{\tau_M}} \leq \beta e^{-\mu t}||X^0||_{L^2_{\tau_M}}, \quad t \geq 0. \tag{3}$$

Note that the exponential stability of the homogeneous system (2) has been explicitly characterized with a spectral condition in (Hale and Verduyn Lunel, 1993, Chapter 9, Theorem 6.1) when the delays are rationally independent [1]. This spectral condition has since been considerably analyzed in the literature (see (Michiels et al., 2009; Henrion and Vyhlídal, 2012; Sipahi et al., 2010; Carvalho, 1996; Damak et al., 2015; Fridman, 2002; Niculescu, 2001; Pepe, 2005)). Alternatively, Lyapunov–Krasovskii functionals with prescribed derivative defined by the so-called Lyapunov delay matrix have also been proposed in (Kharitonov and Zhabko, 2003; Egorov and Mondié, 2014). The properties of this Lyapunov delay matrix and a corresponding complete type Lyapunov–Krasovskii functional has been introduced in (Rocha Campos et al., 2018). This inspired the core of our approach to ISS Lyapunov characterization of system (1).

2.2 Preliminary definitions and properties

In this section, we define the fundamental matrix and the Lyapunov matrix associated with system (2). We also recall some properties that have been shown in (Rocha et al., 2017).

Lemma 2. ((Rocha et al., 2017)). Assume that $\det(\mathrm{Id} - \sum_{k=1}^M A_k) \neq 0$. The $n \times n$ matrix function $K(t)$ defined for all $t \geq 0$ by

$$K(t) = \sum_{k=1}^{M} K(t-\tau_k)A_k = \sum_{k=1}^{M} A_k K(t-\tau_k), \quad t \geq 0, \tag{4}$$

with the initial condition

$$K(\theta) = K_0 = (\sum_{k=1}^{M} A_k - \mathrm{Id})^{-1}, \quad \theta \in [-\tau_M, 0), \tag{5}$$

is called the fundamental matrix of system (2). For any initial condition $X^0 \in C^{\mathrm{pw}}_{\tau_M}$, the response of system (2) is given by

$$X(t) = \sum_{k=1}^{M} D^+ \int_{-\tau_k}^{0} K(t-\theta-\tau_k)A_k X^0(\theta)d\theta. \tag{6}$$

Obviously, the matrix K is perfectly defined when the system (2) is exponentially stable. Formula (6) is known as the **Cauchy formula**. The fundamental matrix $K(t)$ is a piecewise constant function, with discontinuity points defined by

$$t_k = \min_{p_k^1,\dots p_k^m} \{\sum_{j=1}^{M} p_k^j \tau_j \mid \sum_{j=1}^{M} p_k^j \tau_j > t_{k-1},\ p_k^j \in \mathbb{N}\}. \tag{7}$$

We denote the set of discontinuity instants of K as $\mathcal{I}_{\mathcal{K}} = \{t_k\}_{k\in\mathbb{N}}$. For all $t \geq 0$, we define ΔK as

$$\Delta K(t) = K(t^+) - K(t^-). \tag{8}$$

It can be easily verified that $\Delta K(0) = \mathrm{Id}$. Moreover, if the homogeneous system (2) is exponentially stable, then the matrix ΔK exponentially converges to zero. We now define the Lyapunov matrix associated to system (2).

[1] Extending the variable X, it is always possible to rewrite the system in a situation where the delays are rationally independent.

Definition 3. ((Rocha Campos et al., 2018)). Let (2) be exponentially stable. For every $n \times n$ symmetric positive definite matrix W, the Lyapunov matrix

$$U(\tau) = \int_0^{\infty} (K(t) - K_0)^T W K(t+\tau) dt, \tag{9}$$

is well defined for all $\tau \geq -\tau_M$.

The matrix U plays a crucial role in the design of the Lyapunov-Krasovskii functional introduced in (Rocha Campos et al., 2018). Unlike the matrix K, the definition of this functional is only needed on the interval $[-\tau_M, \tau_M]$. Its derivative can be expressed as

$$U'(\tau) = \sum_{k \geq 0} (K^T(t_k - \tau) - K_0^T) W \Delta K(t_k). \tag{10}$$

Due to the discontinuities of K, U' is also discontinuous. We define the derivative's jump discontinuities as

$$\Delta U'(\tau) = U'(\tau^+) - U'(\tau^-), \quad \tau \in [-\tau_M, \tau_M].$$

and it holds for $\tau \in [-\tau_M, \tau_M]$

$$\Delta U'(\tau) = -\sum_{k \geq 0} \Delta K^T(t_k - \tau) W \Delta K(t_k). \tag{11}$$

We recall below some useful properties, proved in (Rocha et al., 2017), of the Lyapunov matrix derivative's jump discontinuities.

Lemma 4. (Rocha et al., 2017) Consider that system (2) is exponentially stable. Then the matrix function $\Delta U'(\tau)$ satisfies

- the Symmetry property
$$\Delta U'(-\tau) = [\Delta U'(\tau)]^T, \tag{12}$$
- the Dynamic property
$$\Delta U'(\tau) = \begin{cases} \sum_{k=1}^{M} \Delta U'(\tau - \tau_k) A_k, \ \tau > 0, \\ \sum_{k=1}^{M} A_k^T \Delta U'(\tau + \tau_k), \ \tau < 0, \end{cases} \tag{13}$$
- and the Generalized algebraic property for all $\tau \geq 0$
$$W \Delta K(\tau) = \sum_{i,j=1}^{M} A_i^T \Delta U'(\tau + \tau_i - \tau_j) A_j - \Delta U'(\tau). \tag{14}$$

It is important to emphasize that when the delays τ_i are not rationally independent the matrix U' may have an infinite number of discontinuities on the interval $[0, \tau_M]$. In what follows, for any $-\tau_M < t_0 < t_1 < \tau_M$, we denote $\mathcal{I}((t_0, t_1))$ the set of discontinuity points of the function U' that belong to (t_0, t_1). We establish the following lemma.

Lemma 5. The set $\mathcal{I}((-\tau_M, \tau_M))$ is countable and $\Delta U'$ is thus equal almost everywhere to the zero function. Moreover, if the homogeneous system (2) is exponentially stable then the quantity $\sum_{\tau_c \in \mathcal{I}((-\tau_M, \tau_M))} ||\Delta U'(\tau_c)||$ is finite.

Proof. Using the definition of $\Delta U'(\tau)$ given in equation (11) and the fact that K is a piecewise constant matrix, we have that $\Delta U'$ is equal to zero on each interval of $(-\tau_M, \tau_M) \cap \mathcal{I}((-\tau_M, \tau_M))^c$. Furthermore, $\tau_c \in (-\tau_M, \tau_M)$ is a discontinuity point if and only if there exists $k \in \mathbb{N}$ such that $t_k - \tau_c \in \mathcal{I}_{\mathcal{K}}$. In other words, τ_c is a discontinuity point if and only if there exist $(k, q) \in \mathbb{N}^2$ such that $t_k - t_q \in (-\tau_M, \tau_M)$ and $\tau_c = t_k - t_q$. As the set $\mathcal{I}_{\mathcal{K}}$ is countable, so is the set $\mathcal{I}((-\tau_M, \tau_M))$. Consequently, we can write

$$\sum_{\tau_c \in \mathcal{I}((-\tau_M, \tau_M))} ||\Delta U'(\tau_c)|| \leq \sum_{q \geq 0} \sum_{k \geq 0} ||\Delta K^T(t_q) W \Delta K(t_k)||.$$

To show that the right-hand side of the previous equation converges, we simply need to prove that the series $\sum_{k \geq 0} ||\Delta K(t_k)||$ converges. Using the exponential convergence of ΔK, there exists $\beta_K \geq 0$ and $\mu_K > 0$ such that for all $t \geq 0$ $||\Delta K(t)|| \leq \beta_K \mathrm{e}^{-\mu_K t}$. Consequently, we have

$$\sum_{k \geq 0} ||\Delta K(t_k)|| \leq \beta_K \sum_{k \geq 0} \mathrm{e}^{-\mu_K t_k}.$$

For all $\ell \in \mathbb{N}$, the number $\ell \tau_1 \in \mathcal{I}_{\mathcal{K}}$. Consequently, there exists $k_\ell \in \mathbb{N}$ such that $t_{k_\ell} = \ell \tau_1$. Similarly, there exists $k_{\ell+1} \in \mathbb{N}$ such that $k_{\ell+1} = (\ell+1)\tau_1$. The number of elements of $\mathcal{I}_{\mathcal{K}}$ that are smaller than $(\ell+1)\tau_1$ is bounded by $(\ell+2)^M$ (maximal number of linear combinations of the τ_i with integer coefficients such that each coefficient is smaller than $(\ell+1)$). Consequently, the number of elements of $\mathcal{I}_{\mathcal{K}}$ that belong to $[\ell\tau_1, (\ell+1)\tau_1)$ is bounded by $(\ell+2)^M$. We denote $\mathcal{I}_\ell$ the set of elements of $\mathcal{I}_{\mathcal{K}}$ that belong to $[\ell\tau_1, (\ell+1)\tau_1)$. We immediately have $\cup_{\ell \in \mathbb{N}} \mathcal{I}_\ell = \mathcal{I}_{\mathcal{K}}$ and Card $(\mathcal{I}_\ell) \leq (\ell+2)^M$. Consequently, we have

$$\begin{aligned} \sum_{k \geq 0} ||\Delta K(t_k)|| &= \sum_{\ell \geq 0} \sum_{t_k \in \mathcal{I}_\ell} ||\Delta K(t_k)|| \\ &\leq \sum_{\ell \geq 0} \sum_{t_k \in \mathcal{I}_\ell} \beta_K \mathrm{e}^{-\mu_K t_k} \leq \sum_{\ell \geq 0} \beta_K \text{ Card } (\mathcal{I}_\ell) \mathrm{e}^{-\mu_K (\ell \delta_1)} \\ &\leq \sum_{\ell \geq 0} \beta_K (\ell+2)^M \mathrm{e}^{-\mu_K \tau_1 \ell}, \end{aligned}$$

which is converging. This concludes the proof. □

Since the set $\mathcal{I}((-\tau_M, \tau_M))$ is countable, we will denote its elements τ_c^q ($q \in \mathbb{N}$). We also define the sequence c_q as

$$c_q = \|\Delta U'(\tau_c^q)\| \tag{15}$$

2.3 Objectives

For the homogeneous system (2), (Rocha Campos et al., 2018) uses the Lyapunov matrix U to design a functional with prescribed derivative, providing that (2) is exponentially stable. Having an explicit Lyapunov functional for the homogeneous equation (2) is an important result as it can pave the path towards stability analysis of the non-homogeneous system (1). More specifically, in this paper, we are interested in the following stability property.

Definition 6. System (1) is said to be ($\mathcal{L}_2$-)*Input-to-State Stable* (ISS) if there exist $R, \lambda, \gamma > 0$ such that, for any piecewise continuous function f, (16)

$$\|X_{[t]}\|_{L^2_{\tau_M}} \leq R e^{-\lambda t} \|X^0\|_{L^2_{\tau_M}} + \gamma \sup_{s \in [0,t]} ||f(s)||^2, \ t \geq 0.$$

Observe that when $f \equiv 0$, this property resumes to the exponential stability of (2), which is thus necessary.

Throughout the next section (Section 3), we will adjust the design proposed in (Rocha Campos et al., 2018) to characterize this ISS property in terms of the existence of a Lyapunov functional, that is equivalent to the L^2-norm of the state. Contrary to (Rocha Campos et al., 2018), where the Lyapunov functional candidate is obtained with the help of the Cauchy formula (following a converse Lyapunov approach), we will directly define the functional of interest and then explicitly compute its time-derivative. Finally, we will show how such a Lyapunov functional can be of interest for the ISS analysis of hyperbolic systems of PDEs (Section 4).

3. LYAPUNOV-KRASOVSKII FUNCTIONAL

To obtain a necessary condition for the Input-to-State Stability of (1), we first assume in the sequel that system (2) is exponentially stable. Correspondingly, we first introduce the functional $v_0(\varphi)$ defined for all $\varphi \in C^{pw}_{\tau_M}$ by

$$v_0(\varphi) = \sum_{i=1}^{M}\sum_{j=1}^{M}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\varphi^T(\xi)A_i^T D_\xi^+ D_\theta^+ \Big(\int_0^\infty K^T(\nu - \xi - \tau_i)WK(\nu-\theta-\tau_j)d\nu\Big)A_j\varphi(\theta)d\theta d\xi, \quad (17)$$

where we have denoted D_ξ^+ and D_θ^+ the Dini derivative with respect to ξ and θ. The integral term $\int_0^\infty K^T(\nu-\xi-\tau_i)WK(\nu-\theta-\tau_j)d\nu$ is well-defined since the fundamental matrix verifies equation (4), which is assumed to be exponentially stable. The functional (17) corresponds to the one given in (Rocha Campos et al., 2018). From the definition of U in equation (9), we obtain

$$\int_0^\infty K^T(\nu-\xi-\tau_i)WK(\nu-\theta-\tau_j)d\nu = \int_0^\infty K_0^T W K(s-\theta-\tau_j)ds + U(-\theta-\tau_j+\xi+\tau_i). \quad (18)$$

Consequently, we have

$$D_\xi^+ D_\theta^+\int_0^\infty K^T(\nu-\xi-\tau_i)WK(\nu-\theta-\tau_j)d\nu = D_\theta^+ D_\xi^+ U(-\theta-\tau_j+\xi+\tau_i). \quad (19)$$

We emphasize that the definition of the functional v_0 requires system (2) to be exponentially stable. We have the following lemma.

Lemma 7. If system (2) is exponentially stable, then there exists $\alpha_1 > 0$, such that for all $\varphi \in C^{pw}_{\tau_M}$

$$0 \le v_0(\varphi) \le \alpha_1 ||\varphi||^2_{L^2_{\tau_M}}. \quad (20)$$

Proof. Define $Y(t,\varphi)$ as the solution of (2) with the initial data φ. Using the Cauchy formula, we can rewrite $v_0(\varphi)$ as an improper integral

$$v_0(\varphi) = \int_0^\infty Y^T(t)WY(t)dt,$$

which is well-defined since Y exponentially converges to zero. Thus $v_0(\varphi) \ge 0$, as W is definite positive. Consider $N \in \mathbb{N}$ and define $g(N) = \int_0^{N\tau_M} Y^T(t)WY(t)dt$. We have

$$g(N) \le ||W|| \sum_{k=0}^{N-1}\int_{-\tau_M}^{0} ||Y(\nu+(k+1)\tau_M)||^2 d\nu \le ||W||\sum_{k=0}^{N-1}\beta^2 \mathrm{e}^{-2\mu(k+1)\tau_M}||\varphi||^2_{L^2_{\tau_M}}$$

where the constants β and μ are defined in (3). This implies the expected result by taking $N \to \infty$. □

The next lemma gives the expression of the time-derivative of $v_0(X_{[t]})$, when $X_{[t]}$ is the solution of equation (1).

Lemma 8. Consider the functional v_0 defined by equation (17) and $X_{[t]}$ the solution of equation (1). Assume that system (2) is exponentially stable. Then, for all $t \ge 0$ we have

$$D^+v_0(X_{[t]}) = -X^T(t)WX(t) - 2X^T(t)\Delta U'(0)f(t) + f^T(t)\Delta U'(0)f(t) - 2\sum_{i=1}^{M}\sum_{\tau_c\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_k-\tau_i)A_i^T\Delta U'(\tau_c)f(t). \quad (21)$$

Note that the expression given by equation (21) is well defined due to Lemma 5.

Proof. Performing the change of variable $\tau = \xi - \tau_j + \tau_i - \theta$, equation (17) rewrites

$$v_0(X_{[t]}) = -\sum_{i=1}^{M}\sum_{j=1}^{M}\int_{-\tau_i}^{0}\int_{\xi+\tau_i-\tau_j}^{\xi+\tau_i} X^T(t+\xi)A_i^T U''(\tau) A_jX(t-\tau+\xi+\tau_i-\tau_j)d\tau d\xi, \quad (22)$$

where, following (Rocha Campos et al., 2018), $U''(\tau)$ is the second-order derivative of the function U. Observe that this second derivative must be understood in the sense of distributions as the function U' may be discontinuous. Let us denote $v_i^j(X_{[t]}) = \int_{-\tau_i}^0\int_{\xi+\tau_i-\tau_j}^{\xi+\tau_i} X^T(t+\xi)A_i^TU''(\tau)A_jX(t-\tau+\xi+\tau_i-\tau_j)d\tau d\xi$. The function v_0 can be written as $v(X_{[t]}) = \sum_{k=1}^3 \gamma_k(X_{[t]})$, where

$$\gamma_1(X_{[t]}) = -\sum_{i=1}^{M} v_i^i(X_{[t]}),\ \gamma_2(X_{[t]}) = -\sum_{i=1}^{M}\sum_{j=i+1}^{M} v_i^j(X_{[t]}),$$
$$\gamma_3(X_{[t]}) = -\sum_{i=1}^{M}\sum_{j=1}^{i-1} v_i^j(X_{[t]}).$$

Using the fact that $\Delta U'$ is equal to zero almost everywhere, that is outside the set $\mathcal{I}((-\tau_M,\tau_M))$ and Fubini's theorem, we obtain

$$\gamma_2(X_{[t]}) = -\sum_{1\le i<j\le M}\Big(\sum_{\tau_c\in\mathcal{I}((-\tau_j,\tau_i-\tau_j))}\int_{-\tau_i}^{\tau_c-\tau_i+\tau_j} L_{ij}(t,\xi,\tau_c)d\xi + \sum_{\tau_c\in\mathcal{I}((\tau_i-\tau_j,0))}\int_{-\tau_i}^{0}L_{ij}(t,\xi,\tau_c)d\xi + \sum_{\tau_c\in\mathcal{I}((0,\tau_i))}\int_{\tau_c-\tau_i}^{0}L_{ij}(t,\xi,\tau_c)d\xi + \int_{-\tau_i}^{0}(L_{ij}(t,\xi,\tau_i-\tau_j)+L_{ij}(t,\xi,0))d\xi\Big), \quad (23)$$

where

$$L_{ij}(t,\xi,\tau_c) = X^T(t+\xi)A_i^T\Delta U'(\tau_c)A_jX(t+\tau_i-\tau_j-\tau_c+\xi).$$

Noticing that $\frac{\partial}{\partial t}L_{ij}(t,\xi,\tau_c) = \frac{\partial}{\partial \xi}L_{ij}(t,\xi,\tau_c)$, it is straightforward to compute the time derivative of γ_2. We obtain after simplification

$$D^+\gamma_2(X_{[t]}) = -\sum_{1\le i<j\le M}\Big(\sum_{\tau_c\in\mathcal{I}((-\tau_j,\tau_i-\tau_j))} L_{ij}(t,\tau_c-\tau_i+\tau_j,\tau_c) - \sum_{\tau_c\in\mathcal{I}((-\tau_j,0))}L_{ij}(t,-\tau_i,\tau_c) + \sum_{\tau_c\in\mathcal{I}((\tau_i-\tau_j,\tau_i))}L_{ij}(t,0,\tau_c) - \sum_{\tau_c\in\mathcal{I}((0,\tau_i))}L_{ij}(t,\tau_c-\tau_i,\tau_c) + L_{ij}(t,0,\tau_i-\tau_j) - L_{ij}(t,-\tau_i,0)\Big).$$

Similar computations can be performed to obtain the time derivatives of γ_1 and γ_3. Summing the different terms, we obtain

$$D^+v_0(X_{[t]}) = -\sum_{1\le i,j\le M}\Big(\sum_{\tau_c\in\mathcal{I}((-\tau_j,\tau_i-\tau_j))} L_{ij}(t,\tau_c-\tau_i+\tau_j,\tau_c)$$
$$-\sum_{\tau_c\in\mathcal{I}((-\tau_j,0))} L_{ij}(t,-\tau_i,\tau_c) + \sum_{\tau_c\in\mathcal{I}((\tau_i-\tau_j,\tau_i))} L_{ij}(t,0,\tau_c) \quad (24)$$
$$-\sum_{\tau_c\in\mathcal{I}((0,\tau_i))} L_{ij}(t,\tau_c-\tau_i,\tau_c) + L_{ij}(t,0,\tau_i-\tau_j) - L_{ij}(t,-\tau_i,0)\Big).$$

Consider the first term of this sum and define $\tau_k = \tau_c + \tau_j$. Using the definitions of the discontinuity points for $\Delta U'$ and the dynamic property of $\Delta U'$ (equation (13)), it rewrites

$$\sum_{i,j=1}^{M}\sum_{\tau_c\in\mathcal{I}((-\tau_j,\tau_i-\tau_j))} L_{ij}(t,\tau_c-\tau_i+\tau_j,\tau_c)$$
$$= \sum_{i,j=1}^{M}\sum_{\tau_k\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_k-\tau_i)A_i^T\Delta U'(\tau_k-\tau_j)A_jX(t),$$
$$\overset{\text{with eq. (13)}}{=} \sum_{i=1}^{M}\sum_{\tau_k\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_k-\tau_i)A_i^T\Delta U'(\tau_k)X(t)$$
$$\overset{\text{with eq. (1)}}{=} \sum_{i=1}^{M}\sum_{\tau_k\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_k-\tau_i)A_i^T\Delta U'(\tau_k) \times\left(\sum_{j=1}^{M}A_jX(t-\tau_j)+f(t)\right)$$
$$= \sum_{i=1}^{M}\sum_{\tau_k\in\mathcal{I}((0,\tau_i))} (\sum_{j=1}^{M}(L_{ij}(t,\tau_k-\tau_i,\tau_k)+M_i(\tau_k)f(t)),$$

where $M_i(\tau_k) = X^T(t+\tau_k-\tau_i)A_i^T\Delta U'(\tau_k)$. The first part of this expression corresponds to the fourth term of (25). Similarly, using $\tau_k = \tau_c - \tau_i$ and (13) we have

$$\sum_{i,j=1}^{M}\sum_{\tau_c\in\mathcal{I}((\tau_i-\tau_j,\tau_i))} L_{ij}(t,0,\tau_c)$$
$$=\sum_{i,j=1}^{M}\sum_{\tau_k\in\mathcal{I}((-\tau_j,0))} X^T(t)A_i^T\Delta U'(\tau_k+\tau_i)A_jX(t-\tau_k-\tau_j),$$
$$=\sum_{j=1}^{M}\sum_{\tau_k\in\mathcal{I}((-\tau_j,0))} X^T(t)\Delta U'(\tau_k)A_jX(t-\tau_k-\tau_j)$$
$$=\sum_{j=1}^{M}\sum_{\tau_k\in\mathcal{I}((-\tau_j,0))} (\sum_{i=1}^{M}A_iX(t-\tau_i)+f(t))^T\Delta U'(\tau_k) \cdot A_jX(t-\tau_k-\tau_j)$$
$$=\sum_{j=1}^{M}\sum_{\tau_k\in\mathcal{I}((-\tau_j,0))} (\sum_{i=1}^{M}L_{ij}(t,-\tau_i,\tau_k)+M_j(-\tau_k)f(t)),$$

since, due to the symmetry property (12), we have $f^T(t)\Delta U'(\tau_k)A_jX(t-\tau_k-\tau_j) = X^T(t-\tau_k-\tau_j)A_j \Delta U'(-\tau_k)f(t)$. Consequently, we obtain

$$D^+v_0(X_{[t]}) = -\sum_{1\le i,j\le M} X^T(t)A_i^T\Delta U'(\tau_i-\tau_j)A_jX(t)$$
$$+\sum_{1\le i,j\le M} X^T(t-\tau_i)A_i^T\Delta U'(0)A_jX(t-\tau_j)$$
$$-2\sum_{i=1}^{M}\sum_{\tau_c\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_c-\tau_i)A_i^T\Delta U'(\tau_c)f(t). \quad (25)$$

Using equation (1), we have

$$D^+v_0(X_{[t]}) = -\sum_{1\le i,j\le M} X^T(t)A_i^T\Delta U'(\tau_i-\tau_j)A_jX(t)$$
$$+(X(t)-f(t))^T\Delta U'(0)(X(t)-f(t))$$
$$-2\sum_{i=1}^{M}\sum_{\tau_c\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_c-\tau_i)A_i^T\Delta U'(\tau_c)f(t). \quad (26)$$

Finally, using the generalized algebraic property (14) and the fact that $\Delta K(0) = \mathrm{Id}$, we obtain

$$D^+v_0(X_{[t]}) = -X^T(t)WX(t) - X^T(t)\Delta U'(0)f(t)$$
$$-f^T(t)\Delta U'(0)X(t) + f^T(t)\Delta U'(0)f(t)$$
$$-2\sum_{i=1}^{M}\sum_{\tau_c\in\mathcal{I}((0,\tau_i))} X^T(t+\tau_c-\tau_i)A_i^T\Delta U'(\tau_c)f(t), \quad (27)$$

which is the expected result. □

In the absence of the exogenous signal f, we have the negativity of the time-derivative of $v_0(X_{[t]})$. However, the functional v_0 is not equivalent to the L^2 norm. Moreover, we wish to obtain a strict Lyapunov functional, that is, with an exponential decay in the absence of the exogenous signal f. Inspired by (Diagne et al., 2012), we introduce two intermediate functionals defined for $\varphi \in C^{pw}_{\tau_M}$ by

$$\bar{v}_0(\varphi) = \sum_{i=1}^{M}\sum_{j=1}^{M}\int_{-\tau_i}^{0}\int_{-\tau_j}^{0}\varphi^T(\xi)A_i^TD_\xi^+D_\theta^+(\int_0^\infty K^T(\nu$$
$$-\xi-\tau_i)WK(\nu-\theta-\tau_j)d\nu)A_j\varphi(\theta)\mathrm{e}^{\frac{\rho}{2}(\theta+\xi)}d\theta d\xi, \quad (28)$$
$$\tilde{v}_0(\varphi) = \bar{v}_0(\varphi) - v_0(\varphi) \quad (29)$$

where $\rho > 0$ is a tuning parameter that will be defined later. Since $\bar{v}_0(\varphi) = v_0(\mathrm{e}^{\frac{\rho}{2}\cdot}\varphi)$, we have $\bar{v}_0(\varphi) \ge 0$ for all $\varphi \in C^{pw}_{\tau_M}$. The functional $\bar{v}_0$ is introduced to obtain an exponential decay rate.

Lemma 9. Consider $X_{[t]}$ the solution of equation (1) and assume that system (2) is exponentially stable. Then, there exist real parameters $K_1 > 0, K_2 > 0, a > 0, \bar{a} > 0$, a sequence of positive coefficients $\bar{d}_q$ such that the series $\sum_{q\ge0}\bar{d}_q$ converges and a sequence of increasing scalar numbers $\bar{\tau}_q$ with $\bar{\tau}_0 = 0$ (all independent on ρ) which are such that, for all $t \ge 0$ and all $\epsilon > 0$, we have

$$D^+\bar{v}_0(X_{[t]}) \le -\rho\bar{v}_0(X_{[t]}) - X^T(t)WX(t) + (\frac{a}{\epsilon}+\bar{a})||f(t)||^2$$
$$+\ (K_1(1-\mathrm{e}^{-\rho\tau_M})+K_2\epsilon)\sum_q \bar{d}_q||X(t-\bar{\tau}_q)||^2. \quad (30)$$

Proof. The computations done in the proof of Lemma 8 can be adjusted to compute the time derivative of $\tilde{v}_0(X_{[t]})$. The terms $L_{ij}(t,\xi,\tau)$ are replaced by

$$\tilde{L}_{ij}(t,\xi,\tau) = \bar{L}_{ij}(t,\xi,\tau) - L_{ij}(t,\xi,\tau) \quad (31)$$

with $\bar{L}_{ij}(t,\xi,\tau) = L_{ij}(t,\xi,\tau)\mathrm{e}^{\frac{\rho}{2}(2\xi-\tau+\tau_i-\tau_j)}$. Thus, taking the time derivative and using the fact that $\frac{\partial}{\partial\xi}\tilde{L}_{ij}(t,\xi,\tau) = \frac{\partial}{\partial t}\tilde{L}_{ij}(t,\xi,\tau) + \rho\bar{L}_{ij}(t,\xi,\tau)$, we obtain

$$D^+\tilde{v}_0(X_{[t]}) = -\sum_{1\le i,j\le M}\Big(-\tilde{L}_{ij}(t,-\tau_i,0) + \tilde{L}_{ij}(t,0,\tau_i$$
$$-\tau_j) - \sum_{\tau_c\in\mathcal{I}((-\tau_j,0))}\tilde{L}_{ij}(t,-\tau_i,\tau_c) + \sum_{\tau_c\in\mathcal{I}((\tau_i-\tau_j,\tau_i))}\tilde{L}_{ij}(t,0,\tau_c)$$

$$+\sum_{\tau_c\in\mathcal{I}((-\tau_j,\tau_i-\tau_j))}\tilde{L}_{ij}(t,\tau_c-\tau_i+\tau_j,\tau_c)$$
$$-\sum_{\tau_c\in\mathcal{I}((0,\tau_i))}\tilde{L}_{ij}(t,\tau_c-\tau_i,\tau_c)\Bigg)-\rho\bar{v}_0(X_{[t]}). \quad (32)$$

Consider $1\le i,j\le M$ and the term $\sum_{\tau_c\in\mathcal{I}((0,\tau_i))}\tilde{L}_{ij}(t,\tau_c-\tau_i,\tau_c)$. Using Young's and Cauchy-Schwarz's inequalities, we have

$$\sum_{\tau_c\in\mathcal{I}((0,\tau_i))}\tilde{L}_{ij}(t,\tau_c-\tau_i,\tau_c)\le(1-\mathrm{e}^{-\rho\tau_M})||A_jX(t-\tau_j)||$$
$$||\sum_{\tau_c\in\mathcal{I}((0,\tau_i))}\Delta U'(\tau_c)A_iX(t+\tau_c-\tau_i)||$$
$$\le\ (1-\mathrm{e}^{-\rho\tau_M})||A||^2||X(t-\tau_j)||\sum_{\tau_c\in\mathcal{I}((0,\tau_i))}||\Delta U'(\tau_c)||$$
$$\cdot||X(t+\tau_c-\tau_i)||$$
$$\le\ \frac{1-\mathrm{e}^{-\rho\tau_M}}{2}(K_0||X(t-\tau_j)||^2$$
$$+||A||^2\sum_{\{q\in\mathbb{N},\ \tau_c^q\in(0,\tau_i)\}}c_q||X(t+\tau_c^q-\tau_i)||^2)$$

where $||A||=\max_{1\le i\le M}||A_i||$, $K_0=||A||^2\sum_{\tau_c\in\mathcal{I}((-\tau_M,\tau_M))}||\Delta U'(\tau_c)||$ (which is well defined due to Lemma 5), and where c_q and τ_c^q are defined in equation (15). Performing analogous computations to deal with the other terms of equation (32), we can define a sequence of coefficients $\tilde{d}_q$ such that the series $\sum_{q\ge0}\tilde{d}_q$ converges and an increasing sequence of delays $\tilde{\tau}_q$ with $\bar{\tau}_0=0$ such that (33)

$$D^+\tilde{v}_0(X_{[t]})\le K_1(1-\mathrm{e}^{\rho\tau_M})\sum_{q\ge0}\tilde{d}_q||X(t-\tilde{\tau}_q)||^2-\rho\bar{v}_0(X_{[t]})$$

for a certain K_1, with all parameters independent of ρ.

Let us now turn our attention to $\bar{v}_0$ which satisfies $D^+\bar{v}_0=D^+\tilde{v}_0+D^+v_0$ with $D^+\tilde{v}_0$ satisfying (33) and D^+v_0 given in (21). Using Young's inequality, for every $\epsilon_0>0$, it holds

$$|X^T\Delta U'(0)f(t)|\le\frac{\epsilon}{2}||X||^2+\frac{1}{2\epsilon}||\Delta U'(0)||^2||f||^2. \quad (34)$$

Finally, we can use Young's inequality to bound the last term of equation (21) by

$$\frac{1}{\epsilon}||f||^2+\epsilon||A||\sum_{i=1}^{M}\sum_{q\in\mathbb{N},\tau_c^q\in(0,\tau_i)}c_q||X(t+\tau_c^q-\tau)||^2. \quad (35)$$

We obtain the expected expression by gathering equations (21), (33), (34), and (35) □

Consider a sequence of positive coefficients b_q such that the series $\sum_{q\ge1}b_q$ converges and define, for all $\varphi\in C^{pw}_{\tau_M}$, the functional v_1 as

$$v_1(\varphi)=\bar{v}_0(\varphi)+\sum_{q\ge1}b_q\int_{-\bar{\tau}_q}^{0}\varphi(\nu)^T\varphi(\nu)\mathrm{e}^{\rho\nu}d\nu$$
$$+b\int_{-\tau_M}^{0}\varphi(\nu)^T\varphi(\nu)\mathrm{e}^{\rho\nu}d\nu, \quad (36)$$

where $b>0$. We have the following lemma.

Lemma 10. For all $\varphi\in C^{pw}_{\tau_M}$, $v_1(\varphi)\ge0$. Consider $X_{[t]}$ the solution of equation (1) and assume that system (2) is exponentially stable. Then, the parameters $\rho>0$, $b>0$, $b_q>0$ and $\epsilon>0$ can be chosen such that

$$D^+v_1(X_{[t]})\le-\rho v_1(X_{[t]})+(\frac{a}{\epsilon}+\bar{a})||f(t)||^2$$
$$-be^{-\rho\tau_M}||X(t-\tau_M)||^2, \quad (37)$$

the coefficients $\bar{a}$ and a being defined in the statement of Lemma 9.

Proof. Obviously, $v_1(\varphi)\ge0$, since $\bar{v}_0(\varphi)\ge0$ Taking time derivative and using integrations by parts, we obtain

$$D^+v_1(X_{[t]})=D^+\bar{v}_0(X_{[t]})+\sum_{q\ge0}b_q||X(t)||^2-\sum_{q\ge0}b_q\mathrm{e}^{-\rho\bar{\tau}_q}$$
$$||X(t-\bar{\tau}_q)||^2-\rho\sum_{q\ge1}b_q\int_{-\bar{\tau}_q}^{0}X(t+\nu)^TX(t+\nu)\mathrm{e}^{\rho\nu}d\nu$$
$$+b||X(t)||^2-be^{-\rho\tau_M}||X(t-\tau_M)||^2$$
$$-\rho b\int_{-\tau_M}^{0}X(t+\nu)^TX(t+\nu)e^{\rho\nu}d\nu$$
$$\le-\rho v_1(X_{[t]})+(\frac{a}{\epsilon}+\bar{a})||f(t)||^2-be^{-\rho\tau_M}||X(t-\tau_M)||^2$$
$$-\sum_{q\ge1}(b_q\mathrm{e}^{-\rho\tau_M}-(K_2\epsilon+K_1(1-\mathrm{e}^{-\rho\tau_M}))\bar{d}_q)||X(t-\bar{\tau}_q)||^2$$
$$+(\sum_{q\ge1}b_q+b-w+\bar{d}_0(K_1(1-\mathrm{e}^{-\rho\tau_M})+K_2\epsilon))||X(t)||^2, \quad (38)$$

where $w>0$ is the smallest eigenvalue of the matrix W. We now choose ϵ,ρ, b and b_q, such that

$$b+\sum_{q\ge1}b_q+\bar{d}_0(K_1(1-\mathrm{e}^{-\rho\tau_M})+K_2\epsilon))-w<0, \quad (39)$$
$$\bar{d}_q(K_1(1-\mathrm{e}^{-\rho\tau_M})+K_2\epsilon)-b_q\mathrm{e}^{-\rho\tau_M}<0. \quad (40)$$

These conditions are always feasible as long as the b_q, b, ρ, and ϵ are chosen small enough. Consequently, we obtain the expected result. □

We now establish the existence of quadratic bounds for v_1.

Lemma 11. If system (2) is exponentially stable, then there exist $\alpha_\ell>0,\alpha_u>0$ such that for all $\varphi\in C^{pw}_{\tau_M}$

$$\alpha_\ell||\varphi||^2_{L^2_{\tau_M}}\le v_1(\varphi)\le\alpha_u||\varphi||^2_{L^2_{\tau_M}}. \quad (41)$$

Proof. The second inequality is easy to show using Lemma 7 and the fact that $\bar{v}_0(\varphi)=v_0(\mathrm{e}^{\frac{\rho}{2}\cdot}\varphi)$. Its proof is omitted. The proof of the first inequality is adjusted from (Rocha Campos et al., 2018). We define the functional $\tilde{v}(\varphi)$ such that $\tilde{v}(\varphi)=v_1(\varphi)-\alpha_\ell||\varphi||^2_{L^2_{\tau_M}}$. We define Y_t as the solution of (2) with the initial data φ. We obtain from equation (37)

$$D^+v_1(Y_t)\le-\rho v_1(Y_t)-be^{-\rho\tau_M}||Y(t-\tau_M)||^2, \quad (42)$$

since $f\equiv0$ for solutions of the homogenous equation. Thus, we have

$$D^+\tilde{v}(Y_t)\le-be^{-\rho\tau_M}||Y(t-\tau_M)||^2$$
$$-\alpha_\ell[Y^T(t)Y(t)-Y^T(t-\tau_M)Y(t-\tau_M)]. \quad (43)$$

Choosing $0<\alpha_\ell\le be^{-\rho\tau_M}$, we obtain $D^+\tilde{v}(Y_t)\le0$. Integrating between 0 and T, we have $\tilde{v}(Y_T)\le\tilde{v}(\varphi)$. The exponential stability of Y_T allows taking $T\to\infty$. We can conclude that $v_1(\varphi)\ge\alpha_\ell||\varphi||^2_{L^2_{\tau_M}}$. □

We can now state the main result of this paper, which characterizes the Input-to-State Stability of System (1) with a Lyapunov functional.

Theorem 12. Consider system (1) with the initial data $X^0 \in C^{pw}_{\tau_M}$. Assume that f belongs to $C^{pw}([0,\infty),\mathbb{R}^n)$. The two following statements are equivalent:

(1) the solution to (1) is $\mathcal{L}_2$-ISS;
(2) there exists a quadratic function $v_1 : C^{pw}_{\tau_M} \to \mathbb{R}_+$ such that
   (a) $\exists\, \rho, \sigma > 0 \quad D^+ v_1(X_{[t]}) \leq -\rho v_1(X_{[t]}) + \sigma \|f(t)\|^2$
   (b) $\exists\, \alpha_l, \alpha_u > 0 \quad \forall \varphi \in C^{pw}_{\tau_M}\ \alpha_\ell ||\varphi||^2_{L^2_{\tau_M}} \leq v_1(\varphi) \leq \alpha_u ||\varphi||^2_{L^2_{\tau_M}}$.

Proof. Let us first prove that (1) implies (2). First, let us observe that (16) implies that the homogeneous equation (4) is exponentially stable. Hence, one can consider the functional v_1 as defined in (36). Lemma 10 then guarantees (a) while Lemma 11 gives (b). Now, assume that (2) holds. From (a), using the comparison principle, we obtain

$$v_1(X_{[t]}) \leq \mathrm{e}^{-\rho t} v_1(X^0) + \int_0^t \sigma \mathrm{e}^{\rho(\nu - t)} ||f(\nu)||^2 d\nu.$$

Using (b), we thus have

$$\begin{aligned} ||X_{[t]}||^2_{L^2_{\tau_M}} &\leq \frac{\alpha_u}{\alpha_\ell} \mathrm{e}^{-\rho t} ||X^0||^2_{L^2_{\tau_M}} + \frac{\sigma}{\alpha_\ell} \int_0^t \mathrm{e}^{\rho(\nu - t)} ||f(\nu)||^2 d\nu \\ &\leq \frac{\alpha_u}{\alpha_\ell} \mathrm{e}^{-\rho t} ||X^0||^2_{L^2_{\tau_M}} + \frac{\sigma}{\rho \alpha_\ell} \sup_{s \in [0,t]} ||f(s)||^2. \end{aligned}$$

Taking the square root we obtain the expected result with $R = \sqrt{\frac{\alpha_u}{\alpha_\ell}}$, $\lambda = \rho/2$ and $\gamma = \sqrt{\frac{\sigma}{\rho \alpha_\ell}}$. □

This result constitutes somehow an extension of (Hale and Verduyn Lunel, 1993, Chapter 9, Theorem 6.1), which proved that the asymptotic stability of (2) is equivalent to the ISS of (1) with respect to the exogenous signal f. Notice however that (Hale and Verduyn Lunel, 1993) consider the ISS with respect to the sup-norm of $X_{[t]}$, which is thus a stronger property than the one we are interested in. In the next section, we illustrate how this result is of interest for hyperbolic PDEs of conservation laws.

Remark 13. The numerical evaluation of the ISS gain γ is an important practical question, which requires exploring the numerical implementation of the Lyapunov functional v_1. The main related difficulty is due to the series $\sum_{q \geq 1} b_q \int_{-\bar{\tau}_q}^0 \varphi(\nu)^T \varphi(\nu) \mathrm{e}^{\rho \nu} d\nu$. Moreover, the term v_0 requires computing the function $U''(\tau)$, which is not an easy task in the case of rationally independent delays. Interestingly, the computations become much simpler when the delays are rationally dependent (as the $\Delta U'$ only has a finite number of discontinuities in this case). Thus, for practical use of the Lyapunov v_1 (to design stabilizing control laws, for instance), one could consider a sufficiently good approximation of the Lyapunov matrix U using rationally dependent delays (see (Rocha et al., 2017) for more details).

4. APPLICATION TO THE STABILITY AND INPUT-TO-STATE STABILITY ANALYSIS OF HYPERBOLIC PDES OF CONSERVATION LAWS

In this section, we show how the Lyapunov functional obtained in Section 3 can be used for the stability analysis of linear hyperbolic PDEs of conservation laws, i.e.

$$u_t(t,x) + \Lambda^+ u_x(t,x) = 0, \tag{44}$$
$$v_t(t,x) - \Lambda^- v_x(t,x) = 0, \tag{45}$$

evolving in $\{(t,x) \mid t > 0,\ x \in [0,1]\}$, where $u = (u_1, \ldots, u_n)^T$, $v = (u_1, \ldots, v_m)^T$ (n and m belonging to $\mathbb{N}\backslash\{0\}$) and u_t and u_x denote time- and space-derivatives, with the following linear boundary conditions

$$u(t,0) = Qv(t,0),\ v(t,1) = Ru(t,1) + f(t). \tag{46}$$

The initial conditions (u_0, v_0) is assumed to belong to $H^1([0,1],\mathbb{R})^{n+m}$. Under appropriate compatibility conditions, the system is well-posed (see (Bastin and Coron, 2016)). The matrices $\Lambda^+ = \mathrm{diag}(\lambda_i)$ and $\Lambda^- = \mathrm{diag}(\mu_i)$ are diagonal and represent the transport velocities. We assume $-\mu_m < \ldots < -\mu_1 < 0$, $\lambda_1 < \ldots < \lambda_n$. The matrices Q and R are constant.

Using the method of characteristics, the PDE system (44)-(46) can be rewritten as a difference systems (2) by defining for all $i \in \{1, \ldots n\}$ and $j \in \{1, \ldots m\}$, the time delays $\tau_{ij} = \frac{1}{\lambda_i} + \frac{1}{\mu_j}$.

Theorem 14. ((Auriol and Di Meglio, 2019)). The stability properties of the system (44)-(46) are equivalent to those of the difference system defined for all $1 \leq i \leq m$ by

$$z_i(t) = \sum_{k=1}^n \sum_{\ell=1}^m Q_{ik} R_{k\ell} z_\ell(t - \tau_{k\ell}) + f_i(t)\,, \tag{47}$$

i.e. there exist two constants $C_1 > 0$ and $C_2 > 0$ and a constant $r > 0$ such that for all $t > \tau$,

$$C_1 ||z_{[t]}||_{L^2_r} \leq ||(u,v)||_{L^2([0,1])} \leq C_2 ||z_{[t]}||_{L^2_\tau}. \tag{48}$$

The proof of this theorem can be found in (Auriol and Di Meglio, 2019). The function z corresponds to $v(t,1)$. It is important to emphasize that the difference system (47) and the PDE system (44)-(46) have equivalent stability properties in the sense of (48). However, they are not strictly equivalent as it may be impossible to reconstruct part of the PDE states (initial condition, for instance) from the state z. In that sense, the system (47) can be seen as a comparison system (see (Niculescu, 2001)). Finally, since the PDE system (44)-(46) is well-posed for H^1 initial conditions that verify the compatibility conditions, we have that the function $v(t,1) = z(t)$ is (piecewise) continuous due to Sobolev embedding theorem.

By means of simple manipulations, equation (47) can be expressed as a homogeneous difference equation (2) (the matrices A_i depending on Q and R). Thus, when $f \equiv 0$, we now have an explicit Lyapunov functional (namely the functional v_1 defined by equation (36)) characterizing the system exponential stability of (44)-(46). This functional is more general than the one given in (Coron et al., 2008; Bastin and Coron, 2016), which not only requires the exponential stability of the system but dissipative boundary conditions, i.e. $\inf\{||\Delta \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \Delta^{-1}||,\ \Delta \in \mathcal{D}_{n+m,+}\} < 1$, where $\mathcal{D}_{n+m,+}$ is the set of diagonal matrices of dimension $n+m$ whose elements on the diagonal are positive. An explicit Lyapunov functional can be of specific interest for control purposes. Indeed, several control strategies (as event-triggered controllers) require Lyapunov functionals (see e.g., (Espitia et al., 2016)). Moreover, it also opens some interesting perspectives regarding robustness analysis (see (Auriol et al., 2022b) for a discussion on the interest of having such a general functional to deal with robustness with respect to stochastic delays in the actuation). This will be the purpose of further investigations.

Remark 15. As mentioned in Remark 13, the Lyapunov functional v_1 may be difficult to compute numerically, and the functional given in (Bastin and Coron, 2016) may appear more amenable. However, one could consider approximating the velocities λ_i and μ_i by rational numbers such that the delays $\tau_{k\ell}$ become rationally independent. In that case, the Lyapunov functional v_1 can be easily computed. The stability of the approximated system will imply the stability of the real one (as long as the approximation is precise enough) due to the inherent robustness properties of the system (see (Auriol et al., 2022a)).

5. CONCLUSION

This paper investigated the Input-to-State Stability of Linear Difference Equations with pointwise delays and proved its equivalence with the existence of an ISS Lyapunov functional. We illustrated how this result could be used for the stability analysis of hyperbolic PDEs of conservation laws. Future works should focus on extending this analysis to LDEs including both pointwise and distributed delays, grounding on the recent necessary Lyapunov stability conditions obtained in (Ortiz et al., 2019, 2022) for an integral delay equation. The proposed analysis could then be applied to hyperbolic PDEs of balance laws and extend the previous Lyapunov conditions obtained in (Bastin and Coron, 2016; Bou Saba et al., 2019; Karafyllis and Krstic, 2019). This is a direction for future works.

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