Convergence of Nonlinear Observers on $\mathbb{R}^n$ with a Riemannian Metric

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We want to design an observer, i.e. a dynamical system with the measurements y coming from a real world process as input and with an output $\widehat{x}$ supposed to be an estimate of the variables x involved in a model of the process.

We address this problem with the following models

$$\frac{dx}{dt} = \dot{x} = f(x) \quad \text{for the time evolution of the variable } x \text{ in } \mathbb{R}^n \text{ to be estimated}$$

$$y = h(x) \quad \text{for describing how the measured output } y \text{ in } \mathbb{R}^p \text{ is related to } x$$

⚠ Warning

To ease the presentation, we assume

- the dynamical model is complete[1] with f and h as smooth as needed and everything is "global".
- time independence. Adding time is technically feasible as long as we "know" the time functions.
- h is a submersion ($\text{rank}\left(\frac{\partial h}{\partial x}\right) = \text{dimension } (y)$)

and we pay no attention to the domains where coordinates are defined.

[1] Solutions exist on $]-\infty, +\infty[$

Definitions :

1. Let Ω_f be the set of ω-limit points of the model $\dot{x} = f(x)$, the set

$$\mathcal{Z} = \{(\hat{x}, x) \in \Omega_f \times \Omega_f : \hat{x} = x\}$$

is called asymptotic zero error set.

2. $\mathcal{Z}$ is said globally asymptotically stable for the extended system :

$$\underbrace{\dot{x} = f(x)}_{\text{given model}} \quad , \quad \underbrace{\dot{\hat{x}} = F(\hat{x}, h(x))}_{\text{observer}}$$

if there exists a complete[1] distance δ on $\mathbb{R}^n$, a continuous function γ and a class $\mathcal{KL}$ function β such that, for any $(x, \hat{x})$ the corresponding solution $(X(x,t), \hat{X}((x,\hat{x}),t))$ in $\mathbb{R}^n \times \mathbb{R}^n$ can be maximally extended to $[0, +\infty[$ and satisfy[2]

$$\delta\Big((X(x,t), \hat{X}((x,\hat{x}),t)),\ \mathcal{Z}\Big) \leq \gamma(x,\hat{x})\, \beta\Big(\delta\Big((x,\hat{x}),\ \mathcal{Z}\Big), t\Big) \qquad \forall t \geq 0 \ .$$

[1]Cauchy sequences converge

[2]Because of γ, not necessarily equi-attractive (= convergence depends only on initial distance).

Problem : Given a pair (f, h), design a function $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ such that the asymptotic zero error set $\mathcal{Z}$ is globally asymptotically stable for the extended system :

$$\dot{x} = f(x) \qquad , \qquad \dot{\widehat{x}} = F(\widehat{x}, h(x))$$

We restrict[1] our attention to the case where

- the observer state equals the state estimate $\widehat{x}$
- model state x and observer state $\widehat{x}$ evolve in the same space.

[1] Nonlinear Luenberger observers for instance do not satisfy this restriction.

Preliminary remarks :

- Although primarily Euclidean in this talk, the space, where x lives, is "structured " by the system properties like infinitesimal observability. This is why we may need a non flat metric to define the distance between x and $\widehat{x}$.
  This is different from using the Riemannian metric linked to the system dynamics (e.g. Aghanan-Rouchon, IEEE TAC 2003). But related to Bonnabel, IEEE TAC 2010.
- The case with a constant Riemannian metric is encountered in most publications.
  But it may not exist and it requires specific coordinates.

Plan

A. About differential detectability

B. About geodesic to metric monotonicity of h

C. Differential detectability
+ Geodesic to metric monotonicity of h

D. Conclusions

E. References

F. Complements

A. About differential detectability

A1. Necessary condition

A2. Sufficient condition

A3. Link with infinitesimal detectability/observability

A4. Link with strong differential observability

Proposition [SP-2009]: Assume the system and the observer are such that the asymptotic zero error set $\mathcal{Z}$ is globally asymptotically stable.
Then there exists a smooth symmetric covariant 2-tensor P with non negative values making the vector field f geodesically weakly monotonic tangentially to the level sets of h i.e.[1]

$$v^\top \mathcal{L}_f P(x)\, v \;\leq\; -v^\top P(x)\, v \qquad \forall (x,v) \in \Omega_f \times \mathbb{R}^n \quad \text{such that} \quad \frac{\partial h}{\partial x}(x)\, v = 0$$

where $\mathcal{L}_f P$ denote the Lie derivative in the direction of f of the covariant 2-tensor P

Proposition [AJP-2016]: If $\mathcal{Z}$ is also uniformly (in x) locally (close to $\mathcal{Z}$) exponentially stable and f and F have bounded derivatives, then there exist a continuous function ρ and real numbers $\underline{p} > 0$, $\overline{p}$ and $q > 0$ such that :

$$\begin{aligned} \underline{p} I \;&\leq\; P(x) \;\leq\; \overline{p} I \\ \mathcal{L}_f P(x) \;&\leq\; \rho(x)\, \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) - q\, P(x) \end{aligned} \qquad \forall x \in \Omega_f$$

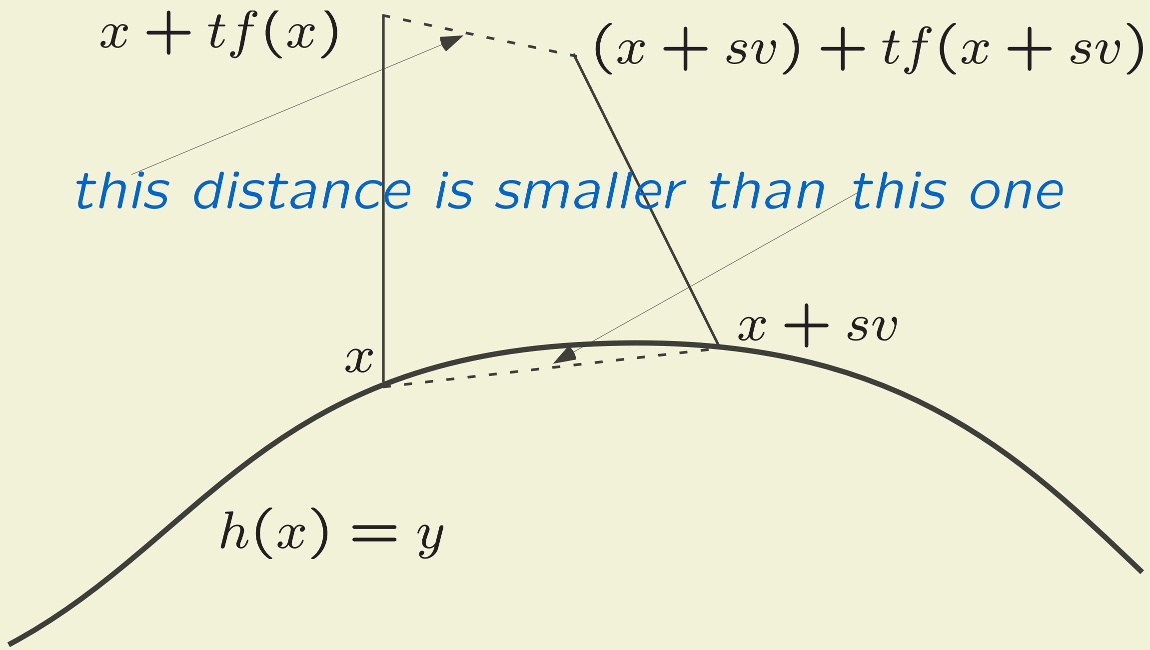
[1] This is a coordinate free condition.

Remark: When P has positive definite values on $\mathbb{R}^n$, it can be used as a metric to define a Riemannian distance.

The distance $d(\widehat{x}, x)$ defined by P makes $\mathbb{R}^n$ a complete Riemannian manifold when

$$\liminf_{|x|\to+\infty} |x|^2 P(x) \;=\; +\infty \;.$$

Meaning of : $$v^\top \mathcal{L}_f P(x)\, v \;\leq\; 0 \qquad \forall v : \frac{\partial h}{\partial x}(x)\, v = 0\ , \quad \forall x \in \Omega_f$$



Contraction at least in the direction tangent to the level sets of h

In the observer jargon . . .

Definitions:

- The system pair (f, h) is said weakly differentially detectable on $\mathbb{R}^n$ if there exists a metric P such that the vector field f is geodesically weakly monotonic tangentially to the level sets of the function h, i.e.

$$v^\top \mathcal{L}_f P(x)\, v \;\leq\; 0 \qquad \forall v : \frac{\partial h}{\partial x}(x) v = 0 \;, \quad \forall x \in \mathbb{R}^n \;.$$

- The system pair (f, h) is said strongly differentially detectable on $\mathbb{R}^n$ if there exists a metric P and a continuous function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\mathcal{L}_f P(x) \;<\; \rho(x) \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \qquad \forall x \in \mathbb{R}^n.$$

Example: Second order system

Let the system be

$$\dot{y} = f_y(z,y) \ , \quad \dot{z} = f_z(z,y)$$

with measured output y.

To obtain the weak differential detectability condition it is sufficient to find a strictly positive solution P_{zz} to

$$= \overbrace{P_{zz}(z,y)}^{\cdot} \quad \text{when} \ \begin{cases} \dot{y} = f_y(z,y) \\ \dot{z} = f_z(z,y) \end{cases}$$

$$\frac{\partial P_{zz}}{\partial y}(z,y) f_y(z,y) + \frac{\partial P_{zz}}{\partial z}(z,y) f_z(z,y) + \left(2\frac{\partial f_z}{\partial z}(z,y) + q\right) P_{zz}(z,y) + 2\frac{\partial f_z}{\partial y}(z,y) P_{yz}(z,y) \ \leq \ 0$$

where we are free to choose $P_{yz}(z,y)$.

Example: Harmonic oscillator with unknown frequency

Let the system be

$$\dot{x}_1 = x_2 , \quad \dot{x}_2 = -x_3 x_1 , \quad \dot{x}_3 = 0 , \quad x_3 > 0$$

with measured output $y = x_1$

With k and ℓ strictly positive real numbers, we choose the metric as

$$P(x) = \begin{pmatrix} 1 + 2\ell k^2 + 4\ell^2 x_1^2 & -2\ell k & 2\ell x_1 \\ -2\ell k & 2\ell & 0 \\ 2\ell x_1 & 0 & 1 \end{pmatrix} .$$

We get

$$\mathcal{L}_f P(x) = \begin{pmatrix} 4\ell k x_3 + 8\ell^2 x_1 x_2 & \star & \star \\ 1 + 2\ell k^2 + 4\ell^2 x_1^2 - 2\ell x_3 & -4\ell k & \star \\ 2\ell k x_1 + 2\ell x_2 & 0 & 0 \end{pmatrix}$$

symmetric

$$\frac{\partial h}{\partial x}(x) v = 0 \quad \Longrightarrow \quad v = (0, v_2, v_3)$$

The evaluation of the Lie derivative of P for a vector v in the kernel of dh gives

$$\begin{pmatrix} v_2 & v_3 \end{pmatrix} \begin{pmatrix} -4\ell k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \end{pmatrix} = -4\ell k v_2^2 \leq 0 \ .$$

Conclusion:

The harmonic oscillator with unknown frequency is weakly differentially detectable for the chosen metric P.

Next question : To obtain a "convergent" observer, is it sufficient to have a metric P making the pair (f, h) strongly differentially detectable, i.e.

$$\mathcal{L}_f P(x) \ < \ \rho(x) \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \qquad \forall x \in \mathbb{R}^n .$$

A. About differential detectability

A1. Necessary condition

A2. Sufficient condition

A3. Link with infinitesimal detectability/observability

A4. Link with strong differential observability

Observer candidate: coordinate invariant See 1 for construction

$$\dot{\hat{x}} = F(\hat{x}, y) = f(\hat{x}) - k_E(x) \underbrace{P(\hat{x})^{-1} \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial \wp}{\partial y_1}(h(\hat{x}), y)^\top}$$

where $(y_1, y_2) \mapsto \wp(y_1, y_2) \geq 0$ is a smooth enough function quantifying how $h(\hat{x})$ is far from $h(x)$ satisfying:

$$\wp(y, y) = 0 \quad , \qquad \frac{\partial^2 \wp}{\partial y_1 \partial y_1}(y, y) > 0$$

Remark: For each y,

$$P(x)^{-1} \frac{\partial h}{\partial x}(x)^\top \frac{\partial \wp}{\partial y_1}(h(x), y)^\top$$

$=$ Riemannian gradient, for the metric P, of the function $x \mapsto \wp(h(x), y)$

$= \text{grad}_P \wp(x, y)$

Proposition [SP-2016]: We assume there exists a complete Riemannian metric P on $\mathbb{R}^n$ for which the pair (f,h) is strongly differentially detectable.

If there exist coordinates such that[1]

$$\frac{1}{c} \leq |P(x)| \leq c\ , \quad |\mathrm{Hess}_P h(x)| \leq c\ , \quad \left|\frac{\partial \wp}{\partial y_1}(h(x_1),h(x_2))\right| \leq c\, d(x_1,x_2)\ ,$$

then there exist strictly positive real numbers $\underline{k}$, ε and $\underline{r}$ such that, with the observer given by

$$\dot{\hat{x}} \;=\; f(\hat{x}) \;-\; k\, \mathrm{grad}_P \wp(x,y)\ ,$$

with $k \geq \underline{k}$, the upper right-hand Dini derivative of the Riemannian distance satisfies

$$\mathfrak{D}^+ d(\hat{x},x) \;\leq\; -\underline{r}\, d(\hat{x},x) \qquad \forall (x,\hat{x})\ :\ d(\hat{x},x) < \frac{\varepsilon}{k}\ .$$

In other words $\mathcal{Z}$ is made uniformly locally exponentially stable.

[1] $\mathrm{Hess}_P h = \frac{1}{2}\mathcal{L}_{\mathrm{grad}_P h} P$ = Riemannian Hessian, for the metric P, of the function $x \mapsto h(x)$

Answer : Yes. Having a metric P making the pair (f, h) strongly differentially detectable is sufficient to obtain a "convergent" observer . . . but locally only.

Next question : How to design the metric P ?

A. About differential detectability

A1. Necessary condition

A2. Sufficient condition

A3. Link with infinitesimal detectability/observability

A4. Link with strong differential observability

To each x, we associate the functions :

$$t \mapsto A_x(t) = \frac{\partial f}{\partial x}(X(x,t)) \quad , \qquad t \mapsto C_x(t) = \frac{\partial h}{\partial x}(X(x,t)) \ .$$

Proposition [SP-2012]: If P is makes the pair (f, g) strongly differentially detectable, then, for each x, there exists a continuous function $t \mapsto K_x(t)$ such that the origin of

$$\dot{\xi} = (A_x(t) - K_x(t)C_x(t))\,\xi$$

is uniformly exponentially stable.

In other words, strong differential detectability $\Rightarrow$ infinitesimal detectability

Conversely . . .

Proposition [SP-2016]: Assume there exist coordinates such that f and h have bounded differential and strictly positive real numbers τ and ϵ such that we have[1] :

$$\int_{-\tau}^{0} \Phi_x(s,0)^\top C_x(s)^\top C_x(s)\Phi_x(s,0)ds \;\geq\; \epsilon\, I \qquad \forall x\ ,$$

= Uniform infinitesimal observability

Let S be a symmetric contravariant 2-tensor satisfying

$$0 \;<\; \underline{s}\, I \;\leq\; S(x) \leq \; \overline{s}\, I\ .$$

Then, there exist a continuous symmetric covariant 2-tensor P with positive definite values which admits a Lie derivative $\mathcal{L}_f P$ satisfying

$$\mathcal{L}_f P(x) \;=\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; P(x)S(x)P(x)$$

= "algebraic" Riccati equation (ARE)

The same results holds for

$$\mathcal{L}_f P(x) \;=\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \lambda\, P(x)$$

[1] Φ_x is the fundamental matrix of $\dot{\xi} = A_x(t)\xi$.

Example: Harmonic oscillator with unknown frequency

It can be obtained using symmetry arguments that, for the harmonic oscillator, a solution to

$$\mathcal{L}_f P(x) \;=\; \frac{\partial h}{\partial x}(x)^\top \frac{\partial h}{\partial x}(x) \;-\; \lambda\, P(x)$$

is

$$P(x) \;=$$

$$\begin{pmatrix} \dfrac{\lambda^2+2x_3}{\lambda(\lambda^2+4x_3)} & , & -\dfrac{1}{(\lambda^2+4x_3)} & , & \dfrac{-\lambda^3 x_1+(\lambda^2-4x_3)x_2}{\lambda^2(\lambda^2+4x_3)^2} \\ -\dfrac{1}{(\lambda^2+4x_3)} & , & \dfrac{2}{\lambda(\lambda^2+4x_3)} & , & \dfrac{(3\lambda^2+4x_3)x_1-4\lambda x_2}{\lambda^2(\lambda^2+4x_3)^2} \\ \dfrac{-\lambda^3 x_1+(\lambda^2-4x_3)x_2}{\lambda^2(\lambda^2+4x_3)^2} & , & \dfrac{(3\lambda^2+4x_3)x_1-4\lambda x_2}{\lambda^2(\lambda^2+4x_3)^2} & , & \left[\begin{array}{r} \dfrac{6\lambda^4+12\lambda^2x_3+16x_3^2}{\lambda^3(\lambda^2+4x_3)^3}x_1^2 \\ -\dfrac{4(5\lambda^2+4x_3)}{\lambda^2(\lambda^2+4x_3)^3}x_1x_2 \\ +\dfrac{4(5\lambda^2+4x_3)}{\lambda^3(\lambda^2+4x_3)^3}x_2^2 \end{array}\right] \end{pmatrix}$$

Numerical scheme to solve the ARE and evaluate P on a grid

Given x in a grid,

1. by integrating backward from x at time 0, up to some sufficiently large time T, compute the solution $t \in [-T, 0] \to X(x,t)$ of

$$\dot{x} = f(x)$$

2. by integrating forward from 0 at time $-T$, up to time 0, compute the the solution $t \in [-T, 0] \to \Pi(t)$ of

$$\dot{\Pi} = -\Pi \frac{\partial f}{\partial x}(X(x,t)) - \frac{\partial f}{\partial x}(X(x,t))^\top \Pi + \frac{\partial h}{\partial x}(X(x,t))^\top \frac{\partial h}{\partial x}(X(x,t)) - \Pi \, S(X(x,t)) \, \Pi$$

Then $\Pi(0)$ is an approximation of $P(x)$

Link with the Extended Kalman filter

- Expressed in coordinates our observer is

$$\dot{\hat{x}} = f(\hat{x}) - k_E(\hat{x})\, P(\hat{x})^{-1}\frac{\partial h}{\partial x}(\hat{x})^\top (y - h(\hat{x}))$$

$$\lim_{t\to 0} \frac{P(z+tf(z)) - P(z)}{t}$$ computed off-line with z as dummy variable.

$$= -P(z)\frac{\partial f}{\partial x}(z) - \frac{\partial f}{\partial x}(z)^\top P(z) + \frac{\partial h}{\partial x}(z)^\top \frac{\partial h}{\partial x}(z) - P(z)S(z)P(z)$$

If the function h were geodesic to metric monotonic[1], we would get semi-global convergence.

- The Extended Kalman filter is

$$\dot{\hat{x}} = f(\hat{x}) - P^{-1}\frac{\partial h}{\partial x}(\hat{x})^\top (y - h(\hat{x}))$$

$$\dot{P} = -P\frac{\partial f}{\partial x}(\hat{x}) - \frac{\partial f}{\partial x}(\hat{x})^\top P + \frac{\partial h}{\partial x}(\hat{x})\frac{\partial h}{\partial x}(\hat{x})^\top - PS(\hat{x})P$$ computed on-line.

Only local convergence is guaranteed.

[1] see below

A. About differential detectability

A1. Necessary condition

A2. Sufficient condition

A3. Link with infinitesimal detectability/observability

A4. Link with strong differential observability

Restriction to the case $p = \dim(h) = 1$

Definition : The system $\dot{x} = f(x) \ , \quad y = h(x)$ is said strongly differentially observable of order n_o if, with $\mathfrak{L}_f h$ be the Lie derivative of h in the direction of f, the function

$$\mathcal{H}_{n_o}(x) = \begin{pmatrix} h(x) \\ \mathfrak{L}_f h(x) \\ \vdots \\ \mathfrak{L}_f^{n_o-1} h(x) \end{pmatrix}$$

is an injective immersion.

Proposition [SP-2016]: If the system is strongly differentially observable of order n_o and there exist coordinates such that we have

$$\underline{h}\, I \;\leq\; \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x) \;\leq\; \overline{h}\, I \quad , \qquad \left|\frac{\partial \mathfrak{L}_f^{n_o} h}{\partial x}(x)\right| \;\leq\; L \left|\frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)\right|$$

then there exists a positive definite symmetric (constant) matrix $\mathcal{P}$ such that

$$P(x) \;=\; \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \mathcal{P}\, \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

makes the pair (f, h) strongly differentially detectable.

Example: Harmonic oscillator with unknown frequency

The harmonic oscillator with unknown frequency is strongly differentially observable of order 4 on the invariant set $\Omega = (\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0,0)\} \times \mathbb{R}_+)$

Indeed, we get
$$\mathcal{H}_4(x) = \begin{pmatrix} x_1 \\ x_2 \\ -x_3\, x_1 \\ -x_3 x_2 \end{pmatrix}$$

$$\frac{\partial \mathcal{H}_4}{\partial x}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \end{pmatrix}$$
implies $\mathcal{H}_4$ is an immersion on Ω.

$$\left.\begin{array}{c} x_{a1}^2 + x_{a2}^2 \neq 0 \\ \text{and} \\ \mathcal{H}_4(x_a) = \mathcal{H}_4(x_b) \end{array}\right\} \Longrightarrow \quad x_a = x_b$$
implies $\mathcal{H}_4$ is injective on Ω.

Conclusion:

There exists a positive definite symmetric (constant) matrix $\mathcal{P}$ such that

$$P(x) = \begin{pmatrix} 1 & 0 & -x_3 & 0 \\ 0 & 1 & 0 & -x_3 \\ 0 & 0 & -x_1 & -x_2 \end{pmatrix} \mathcal{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{pmatrix}$$

makes the harmonic oscillator with unknown frequency strongly differentially detectable

. . . hence a locally convergent observer.

Next question : Under which condition can we get (semi-)global convergence ?

Plan

A. About differential detectability

B. About geodesic to metric monotonicity of h

C. Differential detectability
$+$ Geodesic to metric monotonicity of h

D. Conclusions

E. References

F. Complements

B. About geodesic to metric monotonicity of h

B1. Necessary condition for observer convergence

B2. Sufficient condition for observer convergence

B3. Second fundamental form of h

B4. Design of the metric P

B5. Link with strong differential observability

Definition: Let P be a complete Riemannian metric on $\mathbb{R}^n$.

The observer

$$\dot{\hat{x}} = f(\hat{x}) + \mathfrak{C}(\hat{x}, y)$$

is said to have an infinite gain margin with respect to P if, for any geodesic γ^*, minimal on $[0, \hat{s})$, we have[1]

$$\frac{d\gamma^*}{ds}(s)^\top P(\gamma^*(s))\, \mathfrak{C}\left(\gamma^*(s), h(\gamma^*(0))\right) \;<\; 0 \qquad \forall s \in (0, \hat{s}) \ .$$

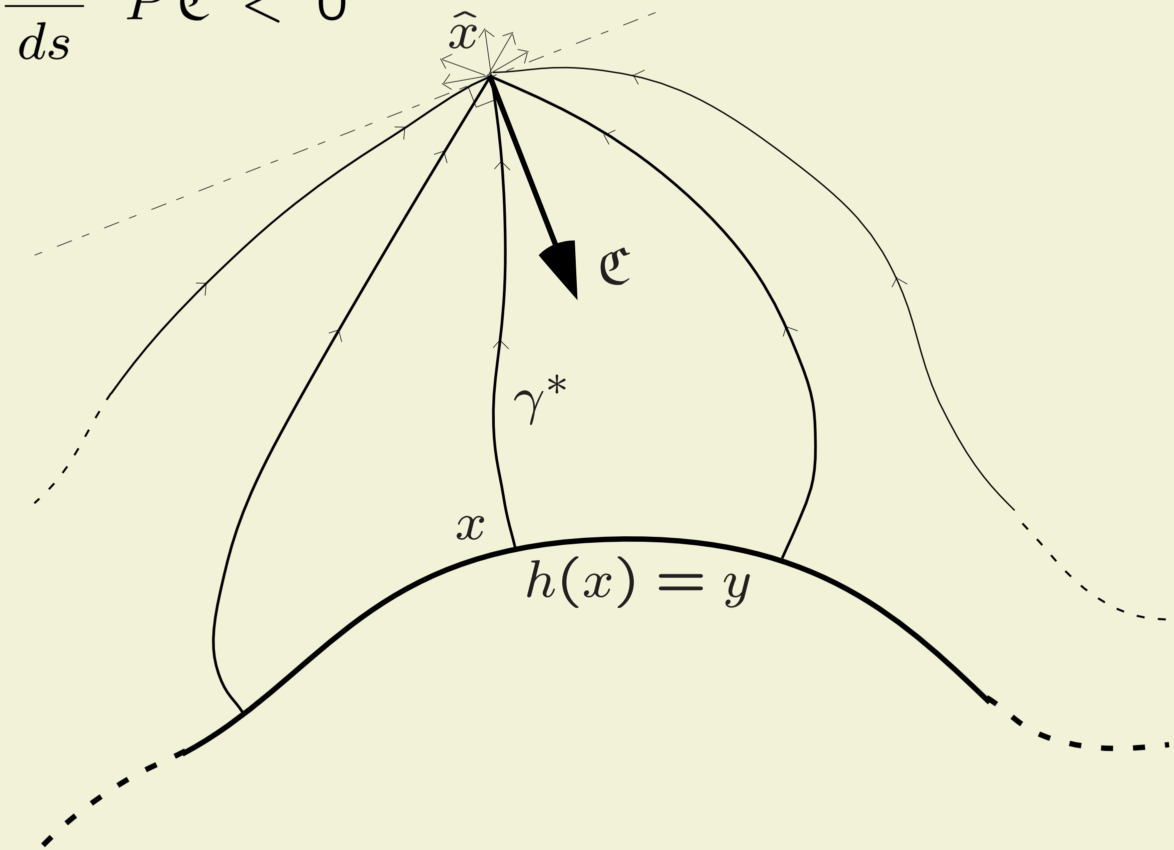
Motivation: if the observer makes[2] $t \mapsto d(\hat{X}((x, \hat{x}), t), X((x), t)$ nonincreasing and the above condition is satisfied, then, for any real number $\ell \geq 1$, the same holds for the observer

$$\dot{\hat{x}} = f(\hat{x}) \;+\; \ell\, \mathfrak{C}(\hat{x}, y)$$

[1] See first variation formula.

[2] d is the Riemannian distance induced by P.

$$\frac{d\gamma^*}{ds}^\top P\,\mathfrak{C} < 0$$



Since our only knowledge is $\hat{x}$ and $h(x) = y$, we need the level set of h $\{z \in \mathbb{R}^n : h(z) = y\}$ to be "seen" from $\hat{x}$ within a cone whose aperture is less than π.

Proposition [SP-2012]: If the observer has an infinite gain margin with respect to P then each level set of h is strongly geodesically convex.

Definition: Let $\wp$ be a function "behaving" like the square of a distance in the y space. A function $h : \mathbb{R}^n \to \mathbb{R}^p$ is said geodesic to metric monotonic if, for any pair (x_a, x_b) satisfying

$$h(x_a) \neq h(x_b)$$

and any minimal geodesic γ^* between $x_a = \gamma^*(s_a)$ and $x_b = \gamma^*(s_b)$, with $s_a \leq s_b$, we have :

$$\frac{d}{ds}\wp(h(\gamma^*(s)), h(\gamma^*(s_a))) > 0 \qquad \forall s \in (s_a, s_b] \ .$$

Proposition [SP-2012]: If the observer correction is a gradient, i.e.

$$\mathfrak{C}(x,y) = P(x)^{-1}\frac{\partial h}{\partial x}(x)^\top \frac{\partial \wp}{\partial y_1}(h(x),y)^\top$$

then infinite gain margin and geodesic to metric monotonicity of h are equivalent properties.

B. About geodesic to metric monotonicity of h

B1. Necessary condition for observer convergence

B2. Sufficient condition for observer convergence

B3. Second fundamental form of h

B4. Design of the metric P

B5. Link with strong differential observability

Proposition [SP-2012] : If P is a complete metric on $\mathbb{R}^n$ for which

1. the pair (f, h) is strongly differentially detectable
2. h is geodesic to metric monotonic for some function $\wp$,

then, for any positive real number E there exists a continuous function $k_E : \mathbb{R}^n \to \mathbb{R}$ such that, with the observer given by

$$\dot{\hat{x}} \;=\; F(\hat{x}, y) \;=\; f(\hat{x}) \;-\; k_E(x)\, P(\hat{x})^{-1} \frac{\partial h}{\partial x}(\hat{x})^\top \frac{\partial \wp}{\partial y_1}(h(\hat{x}), y)^\top \;,$$

the upper right-hand Dini derivative of the Riemannian distance satisfies

$$\mathfrak{D}^+ d(\hat{x}, x) \;\leq\; -\frac{q}{4} d(\hat{x}, x) \qquad \forall (x, \hat{x}) \in \{(x, \hat{x}) \,:\, d(\hat{x}, x) < E\} \;.$$

and it gives an observer with infinite gain margin.

In other words $\mathcal{Z}$ is made semi-globally exponentially stable.

B. About geodesic to metric monotonicity of h

- B1. Necessary condition for observer convergence
- B2. Sufficient conditions for observer convergence
- **B3. Second fundamental form of h**
- B4. Design of the metric P
- B5. Link with strong differential observability

Choice of $\wp$: We choose a complete metric Q for the y manifold such that any piece of geodesic δ is minimizing. Let e denote the corresponding distance (in the y manifold)

Then the function $\wp$ given by

$$\wp(y_1, y_2) = e(y_1, y_2)^2$$

has the required properties, i.e.

$$\wp(y, y) = 0 \quad , \qquad \frac{\partial^2 \wp}{\partial y_1 \partial y_1}(y, y) > 0$$

Choice: the way we quantify how a measurement y_a is far from another measurement y_b does not depend on the states x_a and x_b they are generated by.

Definition: The second fundamental form of the function h is defined as

$$I\!I_{P,Q}[h]^i_{ab}(x) \;=\; \frac{\partial^2 h_i}{\partial x_a \partial x_b}(x) - \Gamma^c_{ab}(x)\frac{\partial h_i}{\partial x_c}(x) + \Delta^i_{jk}(h(x))\frac{\partial h_j}{\partial x_a}(x)\frac{\partial h_k}{\partial x_b}(x)$$

where Γ^c_{ab} and Δ^i_{jk} are the Christoffel symbols associated respectively with P and Q.

Proposition [SP-2020] : If its second fundamental form is zero, then h is geodesic to metric monotonic.

Definition: A function h is called a Riemannian submersion if it is a submersion which satisfies

$$\left(\frac{\partial h}{\partial x}(x)P(x)^{-1}\frac{\partial h}{\partial x}(x)^\top\right)^{-1} = Q(h(x)) \qquad \text{(riem-sub)}$$

Proposition [SP-2020]: The following 2 properties are equivalent:

1. h is a Riemannian submersion the second fundamental form of which is zero.
2. There are everywhere local coordinates $(y, \bar{z})$ for which the expression of the metric is

$$\bar{P}(y,\bar{z}) = \begin{pmatrix} Q(y) & 0 \\ 0 & P_z(\bar{z}) \end{pmatrix} \qquad \text{"diagonalizability"}$$

where P_z does not depend on y.

Reminder: property 1 $\Rightarrow$ h is geodesic to metric monotonic

$\Rightarrow$ semi-global . . . convergence

Proposition [SP-2020]:

a) Conversely, if h is a geodesic to metric monotonic submersion which is Riemannian and $p = dim(h) = 1$, then the "diagonalizability" property 2 above holds.

b) If h is a geodesic to metric monotonic submersion which is not Riemannian, it is made a geodesic to metric monotonic Riemannian submersion when we replace P by

$$P_{mod}(x) \;=\; P(x) + \frac{\partial h}{\partial x}(x)^\top \left[Q(h(x)) - \left(\frac{\partial h}{\partial x}(x) P(x)^{-1} \frac{\partial h}{\partial x}(x)^\top\right)^{-1}\right] \frac{\partial h}{\partial x}(x) \ .$$

Proposition [SP-2016]: (Reduced order observer)

If :

1. the pair (f,h) is strongly differentially detectable with P,

2. h is a Riemannian submersion, the second fundamental form of which is zero,

then there exists everywhere coordinates $x = (y, \bar{z})$ such that, with writing the system dynamics as

$$\dot{y} = f_y(y,\bar{z}) \quad , \qquad \dot{\bar{z}} = f_z(y,\bar{z}) \ ,$$

the flow generated by

$$\dot{\hat{z}} \;=\; f_z(y(t),\hat{z})$$

is a strict contraction for all functions $t \mapsto y(t)$.

$=$ there exists a (globally) convergent reduced order observer, i.e. $\hat{z}$ converges to z.

B. About geodesic to metric monotonicity of h

B1. Necessary condition for observer convergence

B2. Sufficient conditions for observer convergence

B3. Second fundamental form of h

B4. Design of the metric P

B5. Link with strong differential observability

Proposition [SP-2020]:

A – Assume h is a submersion. Let

a) Q be a complete metric for the y manifold such that any piece of geodesic is minimizing,

b) Ξ be an $n-p+q$, with $q \geq 0$, dimensional manifold, equipped with a metric R.

If:

α) $h^\perp : \mathbb{R}^n \to \Xi$ is a function, with rank $n-p$ and such that $(h, h^\perp)$ has rank n,

β) the metric P is given by the sum

$$P(x) = \frac{\partial h}{\partial x}(x)^\top Q(h(x))\frac{\partial h}{\partial x}(x) + \frac{\partial h^\perp}{\partial x}(x)^\top R(h^\perp(x))\frac{\partial h^\perp}{\partial x}(x)$$

γ) $\wp$ is the square of the distance given by Q,

then h is a geodesic to metric monotonic Riemannian submersion

B – Conversely, if h is a Riemannian submersion which is geodesic to metric monotonic and $p = dim(h) = 1$, then P can be decomposed as the sum above.

Remark: If conditions b) and α) hold then Ξ contains a properly embedded submanifold of codimension p which is diffeomorphic to a level set of h.

Example $n = 2\,, \quad p = q = 1\,, \quad h(x_1, x_2) = x_1^2 + x_2^2\,, \qquad \Xi = \mathbb{R}^2$

h is a submersion from the punctured plane $\mathbb{R}^2 \setminus \{0\}$. Its level sets are circles ($= \mathbb{S}^1$).

We look for a function $h^\perp : \mathbb{R}^2 \to \mathbb{R}^2$

1. which has rank 1. Its image is a connected 1-dimensional manifold and therefore diffeomorphic either to an interval of $\mathbb{R}$ or to a circle. Also $\frac{\partial h_1^\perp}{\partial x}(x)$ and $\frac{\partial h_2^\perp}{\partial x}$ are colinear. This implies the existence of two non zero vectors u and v such that
$$\frac{\partial h^\perp}{\partial x}(x) = u(x)v(x)^\top$$
2. such that $(h, h^\perp)$ has rank 2 and therefore satisfying
$$v(x)^\top \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \neq 0 \qquad \forall x \in \mathbb{R}^2 \setminus \{0\}\,.$$

1. and 2. imply $h^\perp(\mathbb{R}^2 \setminus \{0\})$ is diffeomorphic to $\mathbb{S}^1$

A solution is

$$\begin{aligned} h_\alpha^\perp(x_1,x_2) &= \frac{k_\alpha(x_1,x_2)}{\sqrt{k_\alpha(x_1,x_2)^2+k_\beta(x_1,x_2)^2}} \,, \\ h_\beta^\perp(x_1,x_2) &= \frac{k_\beta(x_1,x_2)}{\sqrt{k_\alpha(x_1,x_2)^2+k_\beta(x_1,x_2)^2}} \end{aligned}$$

with $(k_1, k_2) : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ satisfying

$$x_1 \left[k_\beta \frac{\partial k_\alpha}{\partial x_2} - k_\alpha \frac{\partial k_\beta}{\partial x_2}\right] - x_2 \left[k_\beta \frac{\partial k_\alpha}{\partial x_1} - k_\alpha \frac{\partial k_\beta}{\partial x_1}\right] \neq 0 \qquad \forall (x_1,x_2) \in \mathbb{R}^2 \setminus \{0\}$$

Then an expression for P making h geodesic to metric monotonic is as follows with $\chi : \mathbb{R}_{>0} \to \mathbb{R}$ a function with non vanishing derivative, $Q : \mathbb{R} \to \mathbb{R}_{>0}$ and $R : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}_{>0}$ functions with strictly positive values,

$$
\begin{aligned}
&P(x_1, x_2) \\
&\quad = \chi'(x^\top x)^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} Q(\chi(x^\top x)) \begin{pmatrix} x_1 & x_2 \end{pmatrix} \\
&\qquad + \begin{pmatrix} \left[k_\beta(x)\dfrac{\partial k_\alpha}{\partial x_1}(x) - k_\alpha(x)\dfrac{\partial k_\beta}{\partial x_1}(x)\right] \\ \left[k_\beta(x)\dfrac{\partial k_\alpha}{\partial x_2}(x) - k_\alpha(x)\dfrac{\partial k_\beta}{\partial x_2}(x)\right] \end{pmatrix} R(k_\alpha(x), k_\beta(x)) \times \\
&\qquad\qquad \times \begin{pmatrix} \left[k_\beta(x)\dfrac{\partial k_\alpha}{\partial x_1}(x) - k_\alpha(x)\dfrac{\partial k_\beta}{\partial x_1}(x)\right] & \left[k_\beta(x)\dfrac{\partial k_\alpha}{\partial x_2}(x) - k_\alpha(x)\dfrac{\partial k_\beta}{\partial x_2}(x)\right] \end{pmatrix}
\end{aligned}
$$

Example: Harmonic oscillator with unknown frequency

The system is

$$\dot{y} = z_\alpha \,, \quad \dot{z}_\alpha = -z_\beta \, y \,, \quad \dot{z}_\beta = 0 \,, \quad z_\alpha > 0$$

leaving in $\Omega = (\mathbb{R}^2 \times \mathbb{R}_{>0}) \setminus (\{(0,0)\} \times \mathbb{R}_+)$ with $n = 3$, $p = 1$ and and the level sets of the output function being $\mathbb{R} \times \mathbb{R}_{>0}$.

We choose $q = 2$ and $\Xi = \mathbb{R}^2$

With $\chi : \mathbb{R}_{>0} \to \mathbb{R}$ a function with non vanishing derivative, and $h^\perp : \Omega \to \mathbb{R}^2$ such that $(\chi, h^\perp)$ is a diffeomorphism on Ω, $Q : \mathbb{R} \to \mathbb{R}_{>0}$ and $\begin{pmatrix} R_{\alpha\alpha} & R_{\alpha\beta} \\ R_{\alpha\beta} & R_{\beta\beta} \end{pmatrix}$ a function with positive definite values, an expression for P making h geodesic to metric monotonic is

$$P(y, z_\alpha, z_\beta) \;=\; \chi'(y)^2 Q(\chi(y))$$
$$+ \begin{pmatrix} \frac{\partial h_\alpha^\perp}{\partial y}(y, z_\alpha, z_\beta) & \frac{\partial h_\beta^\perp}{\partial y}(y, z_\alpha, z_\beta) \\ \frac{\partial h_\alpha^\perp}{\partial z_\alpha}(y, z_\alpha, z_\beta) & \frac{\partial h_\beta^\perp}{\partial z_\alpha}(y, z_\alpha, z_\beta) \\ \frac{\partial h_\alpha^\perp}{\partial z_\beta}(y, z_\alpha, z_\beta) & \frac{\partial h_\beta^\perp}{\partial z_\beta}(y, z_\alpha, z_\beta) \end{pmatrix} \begin{pmatrix} R_{\alpha\alpha}(\bar{h}^\perp(y, z_\alpha, z_\beta)) & R_{\alpha\beta}(\bar{h}^\perp(y, z_\alpha, z_\beta)) \\ R_{\alpha\beta}(\bar{h}^\perp(y, z_\alpha, z_\beta)) & R_{\beta\beta}(\bar{h}^\perp(y, z_\alpha, z_\beta)) \end{pmatrix} \times$$
$$\times \begin{pmatrix} \frac{\partial h_\alpha^\perp}{\partial y}(y, z_\alpha, z_\beta) & \frac{\partial h_\alpha^\perp}{\partial z_\alpha}(y, z_\alpha, z_\beta) & \frac{\partial h_\alpha^\perp}{\partial z_\beta}(y, z_\alpha, z_\beta) \\ \frac{\partial h_\beta^\perp}{\partial y}(y, z_\alpha, z_\beta) & \frac{\partial h_\beta^\perp}{\partial z_\alpha}(y, z_\alpha, z_\beta) & \frac{\partial h_\beta^\perp}{\partial z_\beta}(y, z_\alpha, z_\beta) \end{pmatrix}$$

B. About geodesic to metric monotonicity of h

B1. Necessary condition for observer convergence

B2. Sufficient conditions for observer convergence

B3. Sufficient condition for geodesic to metric monotonicity

B4. Design of the metric P

B5. Link with strong differential observability

Case $p = dim(h) = 1$

Proposition [SP-2016] : If the system is strongly differentially observable of order n_o, with $n_o = n$ (= the state dimension), then, for the metric given by

$$Q(y) = 1 \quad , \qquad P(x) = \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^\top \mathcal{P} \frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)$$

h is geodesic to metric monotonic.

$\Rightarrow$ If the system is strongly differentially observable of order $n_o = n$,

$$\dot{\hat{x}} = f(\hat{x}) - k_E(\hat{x}) \frac{\partial \mathcal{H}_{n_o}}{\partial x}(\hat{x})^{-1} \mathcal{P}^{-1} \left[\frac{\partial \mathcal{H}_{n_o}}{\partial x}(x)^{-1}\right]^\top \frac{\partial h}{\partial x}(\hat{x})^\top (h(\hat{x}) - y) \ ,$$

is a semi-globally convergent observer.

It is a coordinate free version of the high gain observer.

Plan

A. About differential detectability

B. About geodesic to metric monotonicity of h

C Differential detectability
+ Geodesic to metric monotonicity of h

D. Conclusions

E. References

F. Complements

C. Differential detectability
$+$ Geodesic to metric monotonicity of h

C1. Difficult

C2. Adding inputs

C3. Augmenting the state

Strong differential detectability

$$\sum_c \frac{\partial P_{ab}}{\partial x_c} f_c(x) + P_{ac}(x)\frac{\partial f_c}{\partial x_b}(x) + P_{bc}(x)\frac{\partial f_c}{\partial x_a}(x) \ < \ \rho(x) \sum_j \frac{\partial h_j}{\partial x_a}(x)\frac{\partial h_j}{\partial x_b}(x)$$

f and h, inequality.

Geodesic to metric monotonicity is implied by

$$\frac{\partial^2 h_i}{\partial x_a \partial x_b}(x) - \sum_c \Gamma_{ab}^c(x)\frac{\partial h_i}{\partial x_c}(x) + \sum_{j,k} \Delta_{jk}^i(h(x))\frac{\partial h_j}{\partial x_a}(x)\frac{\partial h_k}{\partial x_b}(x) \ = \ 0$$

where Γ_{ab}^c and Δ_{jk}^i are the Christoffel symbols associated respectively with P and Q.
h only, equality.

Very different nature. We don't know a general design to have both.

Example: Harmonic oscillator with unknown frequency

The model is $\dot{y} = z_\alpha \ , \quad \dot{z}_\alpha = -z_\beta\, y \ , \quad \dot{z}_\beta = 0 \ , \quad z_\alpha > 0$

leaving in $\Omega_\varepsilon = \left\{ (y, z_\alpha, z_\beta) \in \mathbb{R}^3 : \varepsilon < z_\beta y^2 + z_\alpha^2 < \frac{1}{\varepsilon} \ , \ \varepsilon < z_\beta < \frac{1}{\varepsilon} \right\} \ .$

For the sum decomposition of P we choose

$$\bar{z}_\alpha = h_\alpha^\perp(y,z) = z_\alpha - y \quad , \qquad \bar{z}_\beta = h_\beta^\perp(y,z) = z_\beta + \tfrac{1}{2}y^2 + abyz_\alpha$$

$$\chi(y) = y \ , \quad Q(y) = 1 \ , \quad R(\bar{z}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 + a\bar{z}_\alpha^2 \end{pmatrix}$$

This implies h is made geodesic to metric monotonic by[1]

$$P(y, z_\alpha, z_\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & y + abz_\alpha \\ 1 & aby \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 + a\bar{z}_\alpha^2 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ y + abz_\alpha & aby & 1 \end{pmatrix}$$

[1] This metric is not flat.

Then by choosing the real numbers a, b and q satisfying

$$4\left(\frac{1}{\varepsilon}+1\right)^2 \leq b\ , \quad 0 < a\left(\frac{2^{10}[1+8b+2b^2]^2}{\varepsilon^{10}}+(1+b)\right) \leq \varepsilon^2\ , \quad 0 < q \leq \frac{a}{16}\varepsilon^2\ ,$$

P makes the pair (f,h) strongly differentially detectable on Ω_ε.

A globally convergent observer is

$$\overbrace{\begin{pmatrix} \hat{y} \\ \hat{z}_\alpha \\ \hat{z}_\beta \end{pmatrix}}^{\cdot} = \begin{pmatrix} \hat{z}_\alpha \\ -\hat{y}\hat{z}_\beta \\ 0 \end{pmatrix} - k_E(y, z_\alpha, z_\beta)\begin{pmatrix} 1 \\ 1 \\ -\hat{y} - ab(\hat{z}_\alpha + \hat{y}) \end{pmatrix}(\hat{y} - y)$$

C. Differential detectability
$+$ Geodesic to metric monotonicity of h

C1. Difficult

C2. Adding inputs

C3. Augmenting the state

Immersing the model into an input dependent model may help

When $u_y = y$, the solutions of the harmonic oscillator with unknown frequency are solutions of

$$\dot{y} = z_\alpha \quad , \qquad \dot{z}_\alpha = -u_y z_\beta \quad , \qquad \dot{z}_\beta = [u_y - y] z_\alpha$$

input

For this augmented system with u_y as input, the sum formula gives, with u_α another input,

$$P(y, u_y, u_\alpha) = \left(\begin{array}{c|c|c} \begin{array}{l} c + (2a + b^2) - 2y u_y b \\ \quad + y^2(2a + 2u_y^2 + 2u_\alpha^2 - 2u_y u_\alpha) \end{array} & \star & \star \\ \hline -(2a + b^2) + y u_y b & 2a + b^2 & \star \\ \hline \begin{array}{l} -u_y b \\ \quad + y \left(2a + 2u_y^2 + 2u_\alpha^2 - 2u_y u_\alpha\right) \end{array} & u_y b & 2a + 2u_y^2 + 2u_\alpha^2 - 2u_y u_\alpha \end{array}\right)$$

By choosing the real numbers a and b large enough the pair (f,h) can be made strongly differentially detectable for the input dependent model.

This yields the globally convergent observer

$$\begin{aligned}
\dot{\widehat{y}} &= \widehat{z}_\alpha - \frac{k_E}{c}[\widehat{y} - y] \\
\dot{\widehat{z}}_\alpha &= -u_y \widehat{z}_\beta - \frac{k_E}{c}[\widehat{y} - y] \\
\dot{\widehat{z}}_\beta &= [u_y - \widehat{y}]\widehat{z}_\alpha + \tfrac{k_E}{c}\widehat{y}[\widehat{y} - y]
\end{aligned}$$

↖ to be replaced by y for the harmonic oscillator

C. Differential detectability + Geodesic to metric monotonicity of h

C1. Difficult

C2. Adding inputs

C3. Augmenting the state

Dynamic extension may help Well known

We augment the model state x with ξ and consider the augmented model

$$\dot{\overbrace{\begin{pmatrix} x \\ \xi \end{pmatrix}}} = \begin{pmatrix} f_x(x,\xi) \\ f_\xi(x,\xi) \end{pmatrix} \quad , \qquad y_a = h_a(x,\xi) \ ,$$

with f_x, f_ξ and h_a to be chosen under the constraint

$$f_x(x,0) = f(x) \quad , \qquad f_\xi(x,0) = 0 \quad , \qquad h_a(x,0) = h(x) \ .$$

For the harmonic oscillator with unknown frequency we choose

$$\dot{y} = z_\alpha \quad , \quad \dot{z}_\alpha = -y z_\beta + \xi z_\alpha \quad , \quad \dot{z}_\beta = \xi(1 - z_\beta) \quad , \quad \dot{\xi} = -\xi^2 \quad , \quad y_a = y$$

This system is strongly differentially observable of order 4. So

$$P(x) = \left[\frac{\partial}{\partial x}\begin{pmatrix} h(x) \\ \mathfrak{L}_f h(x) \\ \vdots \\ \mathfrak{L}_f^3 h(x) \end{pmatrix}\right]^\top \mathcal{P} \left[\frac{\partial}{\partial x}\begin{pmatrix} h(x) \\ \mathfrak{L}_f h(x) \\ \vdots \\ \mathfrak{L}_f^3 h(x) \end{pmatrix}\right]$$

makes the pair (f, h) strongly differenially detectable and h geodesic to metric monotonic.

Plan

A. About differential detectability

B. About geodesic to metric monotonicity of h

C. Differential detectability
$+$ Geodesic to metric monotonicity of h

D. Conclusions

E. References

F. Complements

We have seen :

To design an observer from a Riemannian metric, we need this metric P to be such that

1. the pair (f, h) is strongly differentially detectable.

2. h is geodesic to metric monotonic.

Point 1: The pair (f, h) is strongly differentially detectable

- is always possible as soon as the first order approximation along any solution is (uniformly) observable or the system is strongly differentially observable.

- P can be obtained as solution of the "algebraic" Riccati equation

$$\sum_c \frac{\partial P_{ab}}{\partial x_c} f_c(x) + P_{ac}(x)\frac{\partial f_c}{\partial x_b}(x) + P_{bc}(x)\frac{\partial f_c}{\partial x_a}(x) + \sum_j \frac{\partial h_j}{\partial x_a}(x)\frac{\partial h_j}{\partial x_b}(x)$$
$$= -\sum_{c,d} P_{ac}(x) S_{cd}(x) P_{db}(x) \qquad (\text{respectively } = -\lambda P_{ab}(x)\ )$$

where S is a symmetric contravariant 2-tensor satisfying $0 < \underline{s}\, I \leq S(x) \leq \overline{s}\, I$.

or as

$$P(x) = \left[\frac{\partial}{\partial x}\begin{pmatrix} h(x) \\ \mathfrak{L}_f h(x) \\ \vdots \\ \mathfrak{L}_f^{n_o - 1} h(x) \end{pmatrix}\right]^\top \mathcal{P} \left[\frac{\partial}{\partial x}\begin{pmatrix} h(x) \\ \mathfrak{L}_f h(x) \\ \vdots \\ \mathfrak{L}_f^{n_o - 1} h(x) \end{pmatrix}\right]$$

Point 2: h is geodesic to metric monotonic

- is necessary for observers of gradient type and with infinite gain margin.
- holds when (respectively is equivalent to, when $p = 1$) the second fundamental form of h is zero.
- is always satisfied when $p = dim(h) = 1$ and the system is strongly differentially observable of order $n_o = n$.
- We can get P as the sum

$$P(x) = \frac{\partial h}{\partial x}(x)^\top Q(h(x)) \frac{\partial h}{\partial x}(x) + \frac{\partial h^\perp}{\partial x}(x)^\top R(h^\perp(x)) \frac{\partial h^\perp}{\partial x}(x)$$

where $h^\perp$ is such that $\mathrm{Rank}(h^\perp) = n - p$ and $\mathrm{Rank}(h, h^\perp) = n$.

The two conditions are satisfied when there are coordinates in which

- the pair (f, h) is strongly differentially detectable with a constant P
- h is linear.

Besides this case, we have no general design to get a metric P satisfying the two conditions.

Open question 1 : Is the property *the system is uniformly infinitesimally observable*[1] a sufficient condition for the existence of a metric making the pair (f, h) strongly differentially detectable and h geodesic to metric monotonic ?

Known answer: "Yes" for the first property

Open question 2 : Does the property h is geodesic to metric monotonic imply its second fundamental form is zero, may be after modifying P into P_{mod}[2] ?

Known answer: "Yes" when $p = 1$

[1] $\int_{-\tau}^{0} \Phi_x(s,0)^\top C_x(s)^\top C_x(s)\Phi_x(s,0)ds \ \geq \ \epsilon\, I$

[2] $P_{mod}(x) \ = \ P(x) + \frac{\partial h}{\partial x}(x)^\top \left[Q(h(x)) - \left(\frac{\partial h}{\partial x}(x)P(x)^{-1}\frac{\partial h}{\partial x}(x)^\top\right)^{-1}\right]\frac{\partial h}{\partial x}(x)$

E. References

Our references

Instrumental references

[AJP-2016] V. Andrieu, B. Jayawardhana, L. Praly. Transverse exponential stability and applications. IEEE Transactions on Automatic Control, Vol. 61, N. 11, November 2016. Pages: 3396-3411

[SP-2009] R. Sanfelice, L. Praly. Nonlinear observer design with an appropriate Riemannian metric. Proceedings of the 48th IEEE Conference on Decision and Control, December 2009

[SP-2012] R. Sanfelice, L. Praly. Convergence of Nonlinear Observers on $\mathbb{R}^n$ with a Riemannian Metric (Part I) IEEE Transactions on Automatic Control, Vol. 57, N. 7, July 2012. Pages: 1709 - 1722
A corrected version can be downloaded from:
`http://cas.ensmp.fr/~praly/Publications/Liste_de_publication.html/ ...`
`...Telechargement/Journaux/2012-IEEE_TAC-Sanfelice-Praly_CORRECTED.pdf`

[SP-2016] R. Sanfelice, L. Praly. Convergence of Nonlinear Observers on $\mathbb{R}^n$ with a Riemannian Metric (Part II). IEEE Transactions on Automatic Control, Vol. 61, N. 10, October 2016. Pages: 2848-2860

[SP-2019] R. Sanfelice, L. Praly. Convergence of Nonlinear Observers on $\mathbb{R}^n$ with a Riemannian Metric (Part III). To be submitted for publication.

E. Instrumental references

[B-1993] P. Bougerol. Kalman filtering with random coefficients and contractions, SIAM J. Control And Optimiz., vol. 31, no. 4, pp. 942959, Jul. 1993.

[E-1925] L. Eisenhart. Riemannian Geometry. Princeton University Press. 1925

[F-1996] A. E. Fischer. Riemannian submersions and the regular interval theorem of morse theory. Annals of Global Analysis and Geometry, 14:263300, 1996.

[I-N-2008] G. Isac and S. Z. Németh. Scalar and asymptotic scalar derivatives : theory and applica- tions. Springer, 4th edition, 2008.

[L-1949] D.C. Lewis. Metric properties of differential equations, American Journal of Mathematics, vol. 71, p. 294-312, 1949.

[O-1966] B. O'Neill. The fundamental equations of a submersion. The Michigan Mathematical Journal, V 13, N. 4, p.459-469, 1966

[V-1970] J. Vilms, Totally geodesic maps. Journal of differential geometry, V. 4, N. 1, p. 73-79, 1970

F. Complements

Construction of the observer with two building materials.

Material 1: Optimization approach

At the current time t, select the estimate $\widehat{x}$ in the set of minimizers $\left(\widehat{x}\,,\,u_{opt}\right)$ of :

$$\mathcal{C}\left(z\,,\,u\,,\,t\right) \;=\; \int_0^t \left[u(s)^\top \Delta(\xi)\,u(s) \;+\; \wp\Big(h(\xi), y\Big)\right]\bigg|_{\xi=X^u(z,t,s)} ds$$

where

- $X^u(z,\tau,s)$ is a (unique) solution of $\dot{x} \;=\; f(x) \;+\; g(x)\,u(t)$ going through z at time t, assumed to be defined at least on $]0,t]$;
- $s \mapsto u(s)$, defined on $]0,t]$, represents modelling errors;
- $\Delta(\xi) \;>\; 0$;
- $(y_1, y_2) \mapsto \wp(y_1, y_2) \geq 0$ is a smooth enough function quantifying how $h(\widehat{x})$ is far from $h(x)$ satisfying

$$\wp(y,y) \;=\; 0 \quad , \qquad \frac{\partial^2 \wp}{\partial y_1 \partial y_1}(y,y) \;>\; 0$$

This is an optimal control problem but in reverse time.

The Hamilton-Jacobi-Bellman equation is

$$\mathsf{HJB}(z,\mathcal{T}) \;=\; 0$$

where

$$\mathsf{HJB}(z,\mathcal{T}) \;=\; \wp\Big(h(z),y(\mathcal{T})\Big) - \frac{\partial \mathcal{V}}{\partial z}(z,\mathcal{T})\, f(z) - \frac{\partial \mathcal{V}}{\partial \mathcal{T}}(z,\mathcal{T}) \\ -\frac{1}{4}\left[\frac{\partial \mathcal{V}}{\partial z}(z,\mathcal{T})g(z)\right]\Delta(z)^{-1}\left[\frac{\partial \mathcal{V}}{\partial z}(z,\mathcal{T})g(z)\right]^{\top},$$

If a solution $\mathcal{V}$ exists satisfying : $\mathcal{V}(z,0)=0$, and if ... and if ... then

$$\mathcal{V}(z,\mathcal{T}) \;=\; \inf_u \int_0^{\mathcal{T}} \left[u(s)^{\top}\Delta(\xi)\,u(s) \;+\; \wp\Big(h(\xi),y\Big)\right]\Big|_{\xi=X^u(z,t,s)} ds \qquad \forall \mathcal{T}\in[0,t]\;.$$

Th current state estimate $\widehat{x}(t)$ is chosen as a minimizer in z of $\mathcal{V}(z,t)$.

Then $\frac{\partial}{\partial t}\left\{\frac{\partial \mathcal{V}}{\partial z}(\widehat{x}(t),t)\right\} = 0$ and $\frac{\partial}{\partial z}\left\{\mathrm{HJB}(z,t)\right\} = 0$

$= 0 \ \forall t$ $= 0 \ \forall t$

give

$$\dot{\widehat{x}} = F(\widehat{x},y) = f(\widehat{x}) - \left(\frac{\partial^2 \mathcal{V}}{\partial z^2}(\widehat{x},t)\right)^{-1} \frac{\partial h}{\partial x}(\widehat{x})^\top \frac{\partial \wp}{\partial y_1}(h(z),y)^\top .$$

Material 2: Contracting flow

Let Π be a complete metric on $\mathbb{R}^n$. Given an input-dependent vector field

$$(x, u) \mapsto \varphi(x, u)$$

the Lie derivative $\mathcal{L}_\varphi \Pi$ of Π in the direction of φ satisfies

$$v^\top \mathcal{L}_\varphi \Pi(x, u)\, v \;=\; \frac{\partial}{\partial x}\left\{v^\top \Pi(x)\, v\right\} \varphi(x, u) + v^\top \Pi(x) \frac{\partial \varphi}{\partial x}(x, u)\, v$$

Lemma [L-1949, ...]: The flow generated by

$$\dot{x} \;=\; \varphi(x, u(t))$$

is a contracting the distance induced by Π if

$$\mathcal{L}_\varphi \Pi(x, u) \;<\; 0$$

Remark : Asymptotic stability of the asymptotic zero error set $\mathcal{Z}$ implies

$$F(x,y) = f(x) + K(x,y) \quad , \qquad K(x,h(x)) = 0 \qquad \forall x \in \Omega_f$$

and therefore

$$\frac{\partial K}{\partial x}(x,h(x)) = -\frac{\partial K}{\partial y}(x,h(x))\frac{\partial h}{\partial x}(x) \qquad \forall x \in \Omega_f$$

Idea: Design K to get $\mathcal{L}_F P(x,y) < 0$

$$\begin{aligned}
\mathcal{L}_F P(x,y) &= \mathcal{L}_f P(x) + \mathcal{L}_K P(x,y) \\
&< \rho(x)\frac{\partial h}{\partial x}(x)\frac{\partial h}{\partial x}(x)^\top + \mathcal{L}_K P(x,y) && \Leftarrow \text{Detectability} \\
\mathcal{L}_F P(x,h(x)) &< \rho(x)\frac{\partial h}{\partial x}(x)^\top\frac{\partial h}{\partial x}(x) - P(x)\frac{\partial K}{\partial y}(x,h(x))\frac{\partial h}{\partial x}(x) && \Leftarrow y = h(x)
\end{aligned}$$

Hence $$\frac{\partial K}{\partial y}(x,h(x)) = -P(x)^{-1}\frac{\partial h}{\partial x}(x)^\top \quad , \qquad R(x,h(x)) \geq \rho(x) I_n$$

Observer candidate

$$\dot{\hat{x}} = F(\hat{x}, y) = f(\hat{x}) - k_E(x) \underbrace{\underbrace{P(\hat{x})^{-1} \frac{\partial h}{\partial x}(\hat{x})^\top}_{\text{contraction}} \underbrace{\frac{\partial \wp}{\partial y_1}(h(\hat{x}), y)^\top}_{\text{optimization}}}$$

gradient, for the metric P, of the function $x \mapsto \wp(h(x), y)$