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Motion planning for a class of partial differential equations with boundary control

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1 Introduction

Motion planning, i.e., the construction of an open-loop control connecting an initial state to a final state, is a fundamental problem of control theory both from a practical and theoretical point of view. For systems governed by *ordinary* differential equations the notion of *flatness* [3, 8] provides a constructive solution to this problem. As noticed in [8], the idea underlying equivalence and flatness –the existence of a one-to-one correspondence between trajectories of systems– is not restricted to *ordinary* differential equations and can be adapted to delay differential systems [9, 4] and partial differential equations [5, 1, 6] with boundary control.

In this paper we study in this spirit the heat equation with one space dimension and control on the boundary. We give an explicit parameterization of the trajectories as a power series in the space variable with coefficients involving time derivatives of the "flat" output. This series is convergent when the flat output is restricted to be a Gevrey function (i.e., a smooth function with a "not too divergent" Taylor expansion). This parameterization provides a new proof of approximate controllability, and above all an explicit open-loop control achieving the desired motion. We then extend some of these results to the general linear diffusion equation.

Our approach is quite different from more "established" theories (see, e.g., [2]) and is more related to older works by for instance Holmgren and Gevrey [12, 7].

2 Gevrey functions

The Taylor expansion of a smooth function is not convergent, unless the function is analytic. The notion of Gevrey order is a way of estimating this divergence.

Definition. A smooth function $t \in [a,b] \mapsto y(t)$ is *Gevrey of order* α if

$$\exists M, R > 0, \forall n \in \mathbb{N}, \quad \sup_{t\in[a,b]} \left|y^{(n)}(t)\right| \leq M\frac{(n!)^\alpha}{R^n}.$$

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By definition, a Gevrey function of order α is also of order β for any $\beta \geq \alpha$. A classical result (the *Cauchy estimates*) asserts that Gevrey functions of order 1 are analytic (entire functions if $\alpha < 1$). Gevrey functions of order $\alpha > 1$ have a divergent Taylor expansion; the larger α, the "more divergent" the Taylor expansion.

Important properties of analytic functions generalize to Gevrey functions of order $\alpha > 1$: the scaling, integration, addition, multiplication and composition of Gevrey functions of order $\alpha > 1$ is of order α [7]. But contrary to analytic functions, functions of order $\alpha > 1$ may be constant on an open set without being constant everywhere. For example the "bump function"

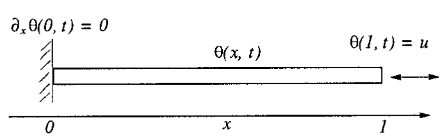
$$\phi_\gamma(t) = \begin{cases} 0 & \text{if } t \notin]0,1[, \\ \exp\left(\dfrac{-1}{((1-t)t)^\gamma}\right) & \text{if } t \in]0,1[. \end{cases}$$

is of order $1 + 1/\gamma$ whatever $\gamma > 0$ [10]. Similarly,

$$\Phi_\gamma(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t \geq 1 \\ \dfrac{\int_0^t \phi_\gamma(\tau)d\tau}{\int_0^1 \phi_\gamma(\tau)d\tau} & \text{if } t \in]0,1[, \end{cases}$$

that will be used for motion planning, has order $1+1/\gamma$.

3 The heat equation is "flat"



Consider the heat equation

$$(1) \qquad \begin{cases} \partial_t\theta(x,t) = \partial_{xx}\theta(x,t), & x \in [0,1] \\ \partial_x\theta(0,t) = 0 \\ \quad \theta(1,t) = u(t), \end{cases}$$

where $\theta(x,t)$ is the temperature and $u(t)$ is the control input. We claim this system is "flat" [3, 8] with

$$y(t) := \theta(0,t)$$

as a "flat" output. In other words, we will show there is (in a certain sense) a $1-1$ correspondence between arbitrary functions $t \mapsto y(t)$ and solutions of (1).

Noticing the "inverse" system

$$\left\{\begin{aligned} \partial_{xx}\theta(x,t) &= \partial_t\theta(x,t) \\ \partial_x\theta(0,t) &= 0 \\ \theta(0,t) &= y(t), \end{aligned}\right. \tag{2}$$

is in Cauchy-Kovalevskaya form, we first seek a formal solution $\theta(x,t) = \sum_{i=0}^{+\infty} a_i(t)\frac{x^i}{i!}$, where the a_i are smooth functions. Using (2), we find

$$\forall i \geq 0, \left\{\begin{aligned} a_{2i}(t) &= y^{(i)}(t) \\ a_{2i+1}(t) &= 0, \end{aligned}\right.$$

so that

$$\theta(x,t) = \sum_{i=0}^{+\infty} y^{(i)}(t)\frac{x^{2i}}{(2i)!} \tag{3}$$

$$u(t) = \sum_{i=0}^{+\infty} \frac{y^{(i)}(t)}{(2i)!}. \tag{4}$$

We give a meaning to this formal solution by restricting $t \mapsto y(t)$ to be Gevrey of a suitable order α. Indeed, the radius ρ of convergence in x of (3) is given by

$$\begin{aligned} \frac{1}{\rho} &= \limsup_{i\to+\infty} \left|\frac{y^{(i)}}{(2i)!}\right|^{\frac{1}{2i}} \\ &\leq \limsup_{i\to+\infty} \left(\frac{M(i!)^\alpha}{R^i(2i)!}\right)^{\frac{1}{2i}} \\ &\leq \limsup_{i\to+\infty} \frac{M^{\frac{1}{2i}}(i!)^{\frac{1}{i}}}{R^{\frac{1}{2}}((2i)!)^{\frac{1}{2i}}}\left((i!)^{\frac{1}{i}}\right)^{\frac{\alpha-2}{2}} \\ &\sim \limsup_{i\to+\infty} \frac{1}{2R^{\frac{1}{2}}}\left(\frac{e}{i}\right)^{1-\frac{\alpha}{2}}, \end{aligned}$$

where we have used the fact that $(n!)^{\frac{1}{n}} \sim n/e$ as an immediate consequence of the Stirling formula. Hence, ρ is greater than $2R^{\frac{1}{2}}$ if $\alpha > 2$ and infinite if $\alpha < 2$.

We have therefore established that (1) is flat in the following sense: any Gevrey function $t \mapsto y(t)$ of order $\alpha < 2$ uniquely defines a smooth trajectory $t \mapsto (\theta(x,t), u(t))$ of (1) which is analytic in x and such that $t \mapsto \theta(0,t)$ is Gevrey of order α. Conversely any trajectory of (1) which is analytic in x and such that $t \mapsto \theta(0,t)$ is Gevrey of order $\alpha < 2$ defines a unique Gevrey function $t \mapsto y(t) := \theta(0,t)$ of order α. For $\alpha = 2$, this 1–1 correspondence holds between trajectories with a radius of convergence > 1 and functions $t \mapsto y(t)$ with $R > 1/4$ (this indeed ensures the convergence of (3) on $[0,1]$).

4 Motion planning

The previous developments provide a simple and explicit solution to the problem of (approximate) motion planning. Assuming the initial temperature profile is

$$\forall x \in [0,1], \quad \theta(x,0) = 0,$$

we want to find an open-loop control $[0,T] \ni t \mapsto u(t)$ such that at time T the final temperature profile is "arbitrary close" to

$$\forall x \in [0,1], \quad \theta(x,T) = \Theta(x), \quad \Theta \in L^2(0,1).$$

Of course Θ does not in general have a convergent Taylor expansion on even powers of x. Nevertheless, as a direct consequence of the Müntz-Szasz theorem (see, e.g., [11, chap. 15]), the set of polynomials of even degree is dense in $C(0,1)$, hence in $L^2(0,1)$. This means that for all $\varepsilon > 0$ there exists a polynomial

$$\Pi(x) = \sum_{i=0}^{n} p_i \frac{x^{2i}}{(2i)!}, \qquad p_i \in \mathbb{R},$$

such that $||\Theta - \Pi||_{L^2} \leq \varepsilon$. On the other hand the function

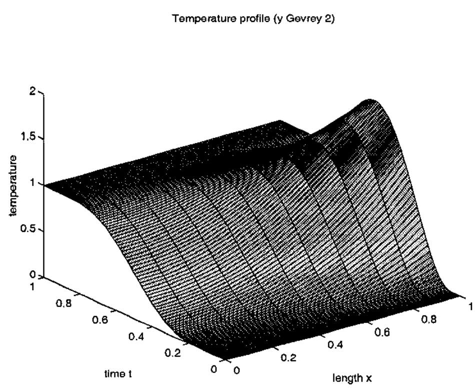
$$y(t) := \left(\sum_{i=0}^{n} p_i \frac{(t-T)^i}{i!}\right) \cdot \Phi_\gamma\left(\frac{t}{T}\right)$$

is Gevrey of order ≤ 2 when $\gamma \geq 1$ (see section 2) and satisfies

$$\begin{aligned} y^{(i)}(0) &= 0, & i &\geq 0 \\ y^{(i)}(T) &= p_i, & i &= 0,\ldots,n \\ y^{(i)}(T) &= 0, & i &> n, \end{aligned}$$

which corresponds to $\theta(x,0) = 0$ and $\theta(x,T) = \Pi(x)$.

Since the solution of (1) starting from a given initial condition is unique, the open-loop control (4) will steer the temperature from 0 at time 0 to Π at time T. We can thus reach at any given time T an arbitrary small neighborhood of Θ. In other words we have proved with elementary and *constructive* arguments that (1) is *approximately controllable for every time T*.



The figure displays the evolution of the temperature with the control generated by $y(t) = \Phi_1(t)$, made to steer the system from the uniform profile $\theta = 0$ at $t = 0$ to the uniform profile $\theta = 1$ at $t = 1$.

5 The linear diffusion equation

We can generalize some of the previous results to the linear diffusion equation

$$(5)\quad \begin{cases} \partial_t\theta = \partial_{xx}\theta + g(x)\partial_x\theta + h(x)\theta, & x\in[0,1] \\ \partial_x\theta(0,t) = 0 \\ \theta(1,t) = u(t), \end{cases}$$

where g and h are analytic functions. We first show that $y(t) := \theta(0,t)$ is again a "flat" output. As before, the "inverse" system

$$(6)\quad \begin{cases} \partial_{xx}\theta = \partial_t\theta - g(x)\partial_x\theta - h(x)\theta \\ \partial_x\theta(0,t) = 0 \\ \theta(0,t) = y(t), \end{cases}$$

is in Cauchy-Kovalevskaya form, and we seek a formal solution $\theta(x,t) = \sum_{i=0}^{+\infty} a_i(t)\frac{x^i}{i!}$, where the a_i are smooth functions.

Theorem. *The formal solution of (6) is convergent when $y(t)$ is Gevrey of order $1 \le \alpha \le 2$.*

Proof. Notice we can assume $g = 0$ by the rescaling

$$\bar\theta(x,t) := \theta(x,t)\ \exp\left(\int_0^x \frac{g(y)}{2}dy\right).$$

Writing $h(x) = \sum_{k\ge 0} h_i\frac{x^i}{i!}$, we easily find the a_k are recursively defined by

$$(7)\quad \begin{cases} a_{k+2}(t) = \dot a_k(t) - \sum_{i=0}^{k}\frac{k!}{i!(k-i)!}h_{k-i}\, a_i(t) \\ a_0(t) = y(t) \\ a_1(t) = 0, \end{cases}$$

and we have to show that $|a_k| \le \frac{\lambda}{\mu^k}k!$ for some $\lambda,\mu > 0$.

The proof is adapted from the classical *method of majorants*: we first replace (lemma 1) the sequence a_k by a "majorizing" sequence A_k such that

$$|a_k| \le \left(A_k(0)\right)^\alpha.$$

This sequence is initialized with $A_0 = A_1 = A$, where

$$\forall t\in[0,r[,\quad A(t) := \frac{m}{1-\frac{t}{r}},$$

with $m, r > 0$. A obviously satisfies

$$A^{(k)} = \frac{m\ k!}{r^k\left(1-\frac{t}{r}\right)^{k+1}}$$

and enjoys a nice differential property (lemma 2).

We then estimate the growth of the A_k in terms of the derivatives of A (lemma 3),

$$A_{2k}, A_{2k+1} \le \frac{(2k)!^{\frac{1}{\alpha}}}{k!}\frac{A^{(k)}}{\rho^k}.$$

We finally conclude $|a_{2k}|, |a_{2k+1}| \le \frac{\tilde m}{(\tilde r\tilde\rho)^k}(2k)!$, which proves the claim. Here and in the sequel we denote $m^\alpha, r^\alpha, \ldots$ by $\tilde m, \tilde r, \ldots$. □

Lemma 1. $\forall\alpha\ge 1$,

$$\begin{cases} A_{k+2} = \dot A_k + \sum_{i=0}^{k}\left(\frac{M}{R^{k-i}}\frac{k!}{i!}\right)^{\frac{1}{\alpha}} A_i \\ A_0 = A \\ A_1 = A \end{cases}$$

is a majorant problem of (7) in the sense that

$$\forall k,n\ge 0,\quad \left|a_k^{(n)}\right| \le \left(A_k^{(n)}(0)\right)^\alpha.$$

Proof. The claim is true at steps 0 and 1 since y is by assumption Gevrey of order α. Assuming it is true till step $k+1$, we prove it is true at step $k+2$; indeed, since h is analytic,

$$\begin{aligned} \left|a_{k+2}^{(n)}\right| &= \left|\dot a_k^{(n)} + \sum_{i=0}^{k}\frac{k!}{i!(k-i)!}a_i^{(n)}h_{k-i}\right| \\ &\le \left|a_k^{(n+1)}\right| + \sum_{i=0}^{k}\frac{M}{R^{k-i}}\frac{k!}{i!}\left|a_i^{(n)}\right| \\ &\le \left(A_k^{(n+1)}(0)\right)^\alpha + \sum_{i=0}^{k}\frac{M}{R^{k-i}}\frac{k!}{i!}\left(A_i^{(n)}(0)\right)^\alpha \\ &\le \left(A_k^{(n+1)}(0) + \sum_{i=0}^{k}\frac{\tilde M}{\tilde R^{k-i}}\left(\frac{k!}{i!}\right)^{\frac{1}{\alpha}}A_i^{(n)}(0)\right)^\alpha \\ &= \left(A_{k+2}^{(n)}(0)\right)^\alpha. \end{aligned}$$

Notice $\sum_q |L_q|^\alpha \le (\sum_q |L_q|)^\alpha$ when $\alpha\ge 1$. □

Lemma 2. $\forall n\ge 0, k\ge j\ge 0$,

$$A^{(j+n)} \le \frac{j!}{k!}r^{k-j}A^{(k+n)}.$$

Proof. As $D(t) := \frac{1}{1-\frac{t}{r}} \le 1$ on $[0,r[$,

$$\begin{aligned} A^{(j+n)} &= \frac{m\ (j+n)!}{r^{j+n}D^{j+n+1}} \\ &\le \frac{m\ (j+n)!}{r^{j+n}D^{k+n+1}} \\ &= \frac{(j+n)!}{(k+n)!}r^{k-j}A^{(k+n)} \\ &\le \frac{j!}{k!}r^{k-j}A^{(k+n)}. \quad \square \end{aligned}$$

Lemma 3. $\forall\alpha\le 2, k\ge 0, n\ge 0$,

$$A_{2k}^{(n)}, A_{2k+1}^{(n)} \le \frac{(2k)!^{\frac{1}{\alpha}}}{k!}\frac{A^{(k+n)}}{\rho^k},$$

where $\frac{1}{\rho} := \max(\frac{r}{\tilde R^2}, 1 + r\tilde M, r\tilde M\tilde R)$.

Proof. The claim is obvious at step 0 since $A_0 = A_1 = A$. Assume then it is true till step k. By definition,

$$A^{(n)}_{2(k+1)} = \dot{A}^{(n)}_{2k} + \sum_{i=0}^{2k} \underbrace{\frac{\tilde{M}}{\tilde{R}^{2k-i}} \left(\frac{(2k)!}{i!}\right)^{\frac{1}{\alpha}}}_{T_i} A^{(n)}_i$$

$$= \dot{A}^{(n)}_{2k} + \sum_{j=0}^{k} T_{2j} A^{(n)}_{2j} + \sum_{j=0}^{k-1} T_{2j+1} A^{(n)}_{2j+1}$$

Using successively the induction assumption, lemma 2 and $\rho \leq \tilde{R}^2/r$, we find

$$\begin{aligned} T_{2j} A^{(n)}_{2j} &\leq \frac{\tilde{M}}{\tilde{R}^{2k-2j}} \frac{(2k)!^{\frac{1}{\alpha}}}{j!} \frac{A^{(j+n)}}{\rho^j} \\ &\leq \frac{\tilde{M} r}{\rho^j} \left(\frac{r}{\tilde{R}^2}\right)^{k-j} \frac{(2k)!^{\frac{1}{\alpha}}}{(k+1)!} A^{(k+1+n)} \\ &\leq \frac{\tilde{M} r}{\rho^k} \frac{(2k)!^{\frac{1}{\alpha}}}{(k+1)!} A^{(k+1+n)}. \end{aligned}$$

Similarly,

$$T_{2j+1} A^{(n)}_{2j+1} \leq \frac{\tilde{M}\tilde{R} r}{\rho^k} \frac{(2k)!^{\frac{1}{\alpha}}}{(k+1)!} A^{(k+1+n)}.$$

On the other hand the induction assumption implies

$$\dot{A}^{(n)}_{2k} \leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!} \frac{r A^{(k+1+n)}}{\rho^k}.$$

Hence,

$$\begin{aligned} A^{(n)}_{2k+2} &\leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!} \frac{A^{(k+1+n)}}{\rho^k} (1 + r\tilde{M} + r\tilde{M}\tilde{R}) \\ &\leq \frac{(2k)!^{\frac{1}{\alpha}}}{k!} \frac{A^{(k+1+n)}}{\rho^{k+1}} \\ &\leq \frac{(2k+2)!^{\frac{1}{\alpha}}}{(k+1)!} \frac{A^{(k+1+n)}}{\rho^{k+1}}. \end{aligned}$$

Notice $k \mapsto \frac{(2k)!^{\frac{1}{\alpha}}}{k!}$ is increasing when $\alpha \leq 2$.

The proof is the same for the odd terms $A^{(n)}_{2k+1}$. □

5.1 Rest-to-rest motion

The general problem of motion planning using flatness is still under study and will be developed elsewhere. Here, we will simply sketch how to use the previous result to steer the diffusion equation from a rest profile to another rest profile.

Clearly, a rest profile $\bar{\theta}$ is characterized by

$$\bar{\theta}(x) = \lambda \theta_0(x),$$

where $\lambda \in \mathbb{R}$ and θ_0 is the solution of

$$\theta_0''(x) + g(x)\theta_0"(x) + \theta_0(x) = 0, \quad \theta_0(0) = 1, \ \theta_0'(0) = 0.$$

This is equivalent to all the $a_k(t)$ in (7) being constant or alternatively to all the derivatives of $y(t)$ being 0.

Hence the open-loop control $u(t) := \sum_{k\geq 0} \frac{a_k(t)}{k!}$ built from the Gevrey function

$$y(t) := \lambda + (\mu - \lambda) \cdot \Phi_\gamma(t/T), \qquad \gamma \geq 1,$$

will steer the system from the rest profile $\theta(x) = \lambda\theta_0(x)$ at time 0 to the rest profile $\theta(x) = \mu\theta_0(x)$ at time T.

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