2015 European Control Conference (ECC)
July 15-17, 2015. Linz, Austria

Finite-Time backstepping boundary stabilization of 3×3 hyperbolic systems

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Abstract— **We consider the problem of boundary stabilization of 3×3 linear first-order hyperbolic systems with one positive and two negative transport speeds by using backstepping. The main result of the paper is to supplement the previous works on how to choose multi-boundary feedback inputs applied on the states corresponding to the negative velocities to obtain finite-time stabilization of the original system in the spatial L^2 sense. Our method is still valid for boundary stabilization of general $n \times n$ hyperbolic system with arbitrary numbers of states traveling in either directions.**

2010 Mathematics Subject Classification. 93D15, 35L04.

Key Words. Finite-Time Stabilization, 3×3 Hyperbolic Systems, Multi-boundary feedback Inputs, Backstepping Transformation

I. Introduction

In this paper, we are concerned with the problem of boundary stabilization for a 3×3 system of first-order hyperbolic linear PDEs with one positive and two negative transport speeds. We consider feedback actuation on the boundary corresponding to the negative velocities.

This problem has been previously investigated by Li and Qin (see [12] and [13]) for even $n \times n$ homogeneous quasilinear hyperbolic systems by using explicit computation of the solution along the characteristic curves in the framework of C^1 norm. Later on, Coron et al. developed a so-called control Lyapunov functions method to analyse the dissipative boundary condition for this kind of nonlinear hyperbolic systems in the content of both C^1 norm and H^2 norm (see [1], [2] and [3]). Recently, Coron and Nguyen showed in [4] that the exponential stability depends strongly on the norm considered, i.e. a previous known sufficient condition for exponential stability with respect to the H^2 norm is not sufficient in the framework of C^1 norm.

For inhomogeneous hyperbolic systems, the study can be dated back to Rauch and Taylor [14] and Russell [15]. Using Lyapunov functions method, Diagne et al. addressed the sufficient boundary conditions for the exponential stability of linear hyperbolic systems of balance laws (see [7]). For the nonlinear cases, Gugat and Herty [9] and Gugat et al.[8] analyzed the boundary feedback stabilization of gas flow in fan-shaped networks governed by isentropic Euler equations. All of these results impose restrictions on the magnitude of the coupling coefficients, which are responsible for potential instabilities.

Coron et al. [5] designed a full-state feedback control law, with actuation on only one end of the domain, which achieves H^2 exponential stability of the closed-loop 2×2 quasilinear hyperbolic system by using backstepping method. Moreover, this method can vanish the corresponding linearized hyperbolic system in finite time. With the same backstepping transformation, Di Meglio et al. [6] showed a feedback controller for the hyperbolic system with n positive and one negative transport speeds, and the feedback actuation only on the state corresponding to the negative velocity in the linear case. These results hold regardless of the (bounded) magnitude of the coupling coefficients. Unfortunately, the method presented both in [5] and [6] can not be extended when several states convecting in the same direction are controlled.

Our contribution in this paper fills the gap in [5] and [6] on how to choose multi-boundary feedback inputs in order to obtain finite-time stabilization of the original system in the spatial L^2 sense. We consider a system of two controlled transport

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equations coupled with one uncontrolled one. This problem has remained unsolved because the traditional backstepping approach of, e.g., [6] yields unsolvable kernel equations. The main contribution of this paper is a novel form of the target system that renders the classical Volterra backstepping transformation well-posed. More precisely, the target system features potentially destabilizing coupling terms that are canceled by a predictor feedback-like boundary condition. Our method can be extended to fully general $n\times n$ hyperbolic system with much more technique involved (see [11]).

The rest of this paper is organized as follows. In Section II, we introduce the system under consideration. In Section III, we detail our choice of target system. In Section IV, we derive the backstepping kernel equations and prove their well-posedness in Section V. Finally, in Section VI we present the main stabilization result.

II. System Description

Consider the following 3 × 3 hyperbolic systems

$$w_t(t,x)+\Lambda w_x(t,x)+\Sigma(x)w(t,x)=0 \quad \text{(II.1)}$$

where, $w=(w_1,w_2,w_3)^T$ is a vector function of (t,x), Λ is a 3×3 diagonal matrix as follows:

$$\Lambda = diag(-\lambda_1,-\lambda_2,\lambda_3), \quad \text{(II.2)}$$

where $\lambda_i(i=1,2,3)$ are positive constants satisfying

$$-\lambda_1<-\lambda_2<0<\lambda_3 \quad \text{(II.3)}$$

The matrix of coupling coefficients $\Sigma(x)$ is a 3×3 matrix with C^0 elements $\sigma_{ij}(x)$, $(i,j=1,2,3)$ and without loss of generality (see Remark 2.1), we assume that $\sigma_{ii}(x)=0$, $(i=1,2,3)$.

The boundary conditions are as follows

$$x=0:\ \ w_3(t,0)=pw_1(t,0)+qw_2(t,0) \quad \text{(II.4)}$$

and

$$\begin{aligned} x=1:\ & w_1(t,1)=h_1(t),\\ & w_2(t,1)=h_2(t). \end{aligned} \quad \text{(II.5)}$$

where $H=(h_1,h_2)^T$ are boundary controls, p and q are constants. Our objective is to design a feedback control law for $H(t)$ in order to ensure that the closed-loop system vanishes in finite time.

Remark 2.1: In general, one can make the following coordinate transformation

$$\widetilde{w}_i(t,x)=\varphi_i(x)w_i(t,x) \quad \text{(II.6)}$$

with $\varphi_i(x) := exp(\int_0^x \frac{\sigma_{ii}(s)}{-\lambda_i}ds)$, $i=1,2$ and $\varphi_3(x):=exp(\int_0^x \frac{\sigma_{33}(s)}{\lambda_3}ds)$. Then the original control system w is transformed into the following system expressed in the new coordinates:

$$\widetilde{w}_t(t,x)+\Lambda\widetilde{w}_x(t,x)+\widetilde{\Sigma}(x)\widetilde{w}(t,x)=0, \quad \text{(II.7)}$$

in which $\widetilde{\Sigma} := (\widetilde{\sigma}_{ij})_{3\times 3} \in (C^0[0,1])^{3\times 3}$ with $\widetilde{\sigma}_{ii}(x)=0(i=1,2,3)$.

III. Target System

As mentioned in the introduction, the main contribution of this paper is a novel form of the target system. More precisely, our approach to design $H(t)$ will be seek a mapping that transforms (II.1), (II.4)–(II.5) into

$$\gamma_t(t,x)+\Lambda\gamma_x(t,x)+G(x)\gamma(t,0)=0 \quad \text{(III.1)}$$

with the boundary conditions

$$x=0:\quad \gamma_3(t,0)=p\gamma_1(t,0)+q\gamma_2(t,0) \quad \text{(III.2)}$$

and

$$\begin{aligned} x=1: & \ \gamma_1(t,1)=0,\\ & \gamma_2(t,1)=f(\gamma_2(t,\cdot)), \end{aligned} \quad \text{(III.3)}$$

where $f(\gamma_2(t,\cdot))$ is chosen as the following predictor-based feedback

$$f(\gamma_2(t,\cdot))=\int_0^{\lambda_2^{-1}} g_{22}(1-\lambda_2\tau)\gamma_2(t,\lambda_2\tau)d\tau - \int_0^{\lambda_2^{-1}}\int_0^{\tau} g_{22}(1-\lambda_2\tau)l(\tau,s)\gamma_2(t,\lambda_2 s)dsd\tau \quad \text{(III.4)}$$

where $l(\cdot,\cdot)$ is implicitly defined by the following Volterra equation of the second kind

$$l(\tau,s)=g_{22}(\lambda_2\tau-\lambda_2 s) - \int_s^{\tau} g_{22}(\lambda_2\tau-\lambda_2\nu)l(\nu,s)d\nu \quad \text{(III.5)}$$

and $G\in(C^0[0,1])^{3\times 3}$ has the following form

$$G(x):=\begin{pmatrix} 0 & 0 & 0\\ g_{21}(x) & g_{22}(x) & 0\\ g_{31}(x) & g_{32}(x) & 0\end{pmatrix}. \quad \text{(III.6)}$$

where the g_{ij} will be determined in § IV. The following Proposition guarantees finite-time convergence of the cascade system (III.1)–(III.5).

Proposition 3.1: Let g_{21}, g_{22}, g_{31} and g_{32} be $C^0([0,1])$ functions. The mixed initial-boundary problem (III.1)–(III.5) with initial condition

$$t=0:\gamma(0,x)=\gamma_0(x), \quad \text{(III.7)}$$

in which $\gamma_0 := (\gamma_{01}, \gamma_{02}, \gamma_{03}) \in (L^2(0,1))^3$ admits a $C^0([0,\infty);(L^2(0,1))^3)$ solution $\gamma = \gamma(t,x)$, which vanishes to equilibrium $\gamma \equiv 0$ in finite time $t \geq t_F$, where t_F is given by

$$t_F = \frac{1}{\lambda_1} + \frac{2}{\lambda_2} + \frac{1}{\lambda_3} \quad \text{(III.8)}$$

Proof. To show finite-time convergence to the origin, one can find the explicit solution of (III.1)-(III.3) as follows. The explicit solution of γ_1, by noting (III.1) and (III.3), is given by

$$\gamma_1(t,x) = \begin{cases} \gamma_{01}(x+\lambda_1 t) & \text{if } t < \frac{1-x}{\lambda_1}, \\ 0 & \text{if } t \geq \frac{1-x}{\lambda_1}. \end{cases} \quad \text{(III.9)}$$

Notice in particular that γ_1 is identically zero for $t \geq \lambda_1^{-1}$. From (III.1) and (III.6), we obtain that $\gamma_2(t,x)$ satisfies the following equation

$$\gamma_{2t}(t,x) - \lambda_2\gamma_{2x}(t,x) + g_{22}(x)\gamma_2(t,0) = 0. \quad \text{(III.10)}$$

Similarly, the explicit expression of $\gamma_2(t,0)$ is given, for $t > \lambda_1^{-1} + \lambda_2^{-1}$, by

$$\gamma_2(t,0) = \gamma_2(t-\lambda_2^{-1},1) - \int_0^{\lambda_2^{-1}} g_{22}(1-\lambda_2\tau)\gamma_2(t-\lambda_2^{-1}+\tau,0)d\tau$$

Thus, choosing the following form for $\gamma_2(t,1)$

$$\gamma_2(t,1) = \int_0^{\lambda_2^{-1}} g_{22}(1-\lambda_2\tau)\gamma_2(t+\tau,0)d\tau \quad \text{(III.11)}$$

ensures

$$\gamma_2(t,0) = 0 \quad \forall t \geq \lambda_1^{-1} + \lambda_2^{-1}. \quad \text{(III.12)}$$

Unfortunately, Equation (III.11) uses future values of $\gamma_2(\cdot,0)$ and is not implementable as such. However, one can obtain the prediction of $\gamma_2(t+\tau,0)$, for $\tau \in [0,\lambda_2^{-1}]$ implicitly defined by characteristic method as follows

$$\gamma_2(t+\tau,0) + \int_0^\tau g_{22}(\lambda_2\tau - \lambda_2 s)\gamma_2(t+s,0)ds = \gamma_2(t,\lambda_2\tau), \quad \text{(III.13)}$$

which is a Volterra equation of the second kind in the unknown $\gamma_2(t+\cdot,0)$ on $[0,\lambda_2^{-1}]$. Since $g_{22}(\cdot) \in C^0[0,1]$, we can see from [10] and [16] that (III.13) has the following solution

$$\gamma_2(t+\tau,0) = \gamma_2(t,\lambda_2\tau) - \int_0^\tau l(\tau,s)\gamma_2(t,\lambda_2 s)ds$$

where the inverse kernel $l(\cdot,\cdot)$ is defined by (III.5), which yields (III.4). Besides, given (III.12), γ_2 satisfies, for $t \geq \lambda_1^{-1}+\lambda_2^{-1}$, the following equation

$$\gamma_{2t}(t,x) - \lambda_2\gamma_{2x}(t,x) = 0, \quad \forall t \geq \lambda_1^{-1} + \lambda_2^{-1} \quad \text{(III.14)}$$

with the boundary

$$\gamma_2(t,1) = \int_0^{\lambda_2^{-1}} g_{22}(1-\lambda_2\tau)\gamma_2(t,\lambda_2\tau)d\tau - \int_0^{\lambda_2^{-1}}\int_0^\tau g_{22}(1-\lambda_2\tau)l(\tau,s)\gamma_2(t,\lambda_2 s)dsd\tau$$

which is identically zero thanks to (III.13). This yields

$$\gamma_2(t,x) \equiv 0 \qquad \forall t \geq \lambda_1^{-1} + 2\lambda_2^{-1}. \quad \text{(III.15)}$$

Similarly, when $t \geq \lambda_1^{-1} + 2\lambda_2^{-1}$, $\gamma_3(t,x)$ satisfies the following equations

$$\gamma_{3t}(t,x) + \lambda_3\gamma_{3x}(t,x) = 0. \quad \text{(III.16)}$$

and the explicit solution of (III.16) and

$$x = 0 : \gamma_3(t,0) = 0, \quad t \geq \lambda_1^{-1} + 2\lambda_2^{-1} \quad \text{(III.17)}$$

is

$$\gamma_3(t,x) = \quad \text{(III.18)}$$

$$\begin{cases} \gamma_3\big(\lambda_1^{-1} + 2\lambda_2^{-1}, x - \lambda_3(t - \lambda_1^{-1} - 2\lambda_2^{-1})\big) \\ \quad \text{if } \lambda_1^{-1} + 2\lambda_2^{-1} \leq t \leq \lambda_1^{-1} + 2\lambda_2^{-1} + \lambda_3^{-1}x, \\ 0 \quad \text{if } t > \lambda_1^{-1} + 2\lambda_2^{-1} + \lambda_3^{-1}x. \end{cases} \quad \text{(III.19)}$$

Thus, when $t \geq \lambda_1^{-1} + 2\lambda_2^{-1} + \lambda_3^{-1}$, we have

$$\gamma_3(t,x) \equiv 0. \quad \text{(III.20)}$$

Thus, one has $\gamma(t,x) \equiv 0$, which concludes the Proof. ■

As will appear in the next two sections, this particular choice of the target system makes the backstepping kernel equations relatively straightforward to solve.

IV. Backstepping transformation and Kernel Equations

To map the original system (II.1) into the target system (III.1), we use the following transformation

$$\gamma(t,x) = w(t,x) - \int_0^x K(x,y)w(t,y)dy \quad \text{(IV.1)}$$

We point out here that this transformation yields that $w(t,0) \equiv \gamma(t,0)$ ($\forall t > 0$), which is crucial to design our feedback law. Utilizing (II.1) and straightforward computations, one can show that

$$\begin{aligned}\gamma_t + \Lambda\gamma_x = \\ &-\int_0^x \big[K_y(x,y)\Lambda + \Lambda K_x(x,y) \\ &\quad - K(x,y)\Sigma(y)\big] w(t,y)dy \\ &+ \big(K(x,x)\Lambda - \Lambda K(x,x) - \Sigma(x)\big) w(t,x) \\ &\quad - K(x,0)\Lambda(0)w(t,0)\end{aligned}$$

The original system (II.1) is mapped into the target system (III.1) if K and G satisfy the following equations

$$g_{21}(x) = -\lambda_1 k_{21}(x,0) + p\lambda_3 k_{23}(x,0) \quad \text{(IV.2)}$$
$$g_{22}(x) = -\lambda_2 k_{22}(x,0) + q\lambda_3 k_{23}(x,0) \quad \text{(IV.3)}$$
$$g_{31}(x) = -\lambda_1 k_{31}(x,0) + p\lambda_3 k_{33}(x,0) \quad \text{(IV.4)}$$
$$g_{32}(x) = -\lambda_2 k_{32}(x,0) + q\lambda_3 k_{33}(x,0) \quad \text{(IV.5)}$$

and

$$\Lambda K_x(x,y) + K_y(x,y)\Lambda - K(x,y)\Sigma(y) = 0 \quad \text{(IV.6)}$$

with the boundary conditions

$$\begin{cases} k_{11}(x,0) = \dfrac{\lambda_3}{\lambda_1} p k_{13}(x,0), \\ k_{12}(x,0) = \dfrac{\lambda_3}{\lambda_2} q k_{13}(x,0), \\ k_{12}(x,x) = \dfrac{\sigma_{12}(x)}{\lambda_1 - \lambda_2}, \\ k_{13}(x,x) = \dfrac{\sigma_{13}(x)}{\lambda_1 + \lambda_3}, \\ k_{21}(x,x) = \dfrac{\sigma_{21}(x)}{\lambda_2 - \lambda_1}, \\ k_{23}(x,x) = \dfrac{\sigma_{23}(x)}{\lambda_2 + \lambda_3}, \\ k_{31}(x,x) = -\dfrac{\sigma_{31}(x)}{\lambda_1 + \lambda_3}, \\ k_{32}(x,x) = -\dfrac{\sigma_{32}(x)}{\lambda_2 + \lambda_3} \end{cases} \quad \text{(IV.7)}$$

In particular, the choice of boundary conditions for $k_{21}(1,\cdot)$, $k_{22}(1,\cdot)$ and $k_{33}(\cdot,0)$ is free. We chose the following structures

$$\begin{cases} k_{21}(1,y) = M_1(y) \\ k_{22}(1,y) = M_2(y) \\ k_{33}(x,0) = M_3(x) \end{cases} \quad \text{(IV.8)}$$

where the $M_i(\cdot)$, $i = 1,2,3$ can be chosen as desired with

$$M_1(1) = \frac{\sigma_{21}(1)}{\lambda_2 - \lambda_1}. \quad \text{(IV.9)}$$

The equations evolve in the triangular domain $\mathcal{T} = \{(x,y) : 0 \le y \le x \le 1\}$. By Theorem 5.1, under the assumption that $M_1(\cdot)$, $M_2(\cdot)$, $M_3(\cdot)$ and $\sigma_{ij}(\cdot)$ are $C^0[0,1]$ and (IV.9) holds, One finds that there is a unique L^∞ solution $K(x,y)$ to (IV.6)-(IV.8), which is continuous for $k_{ij}(\cdot,\cdot)$ for any $i \neq 1$.

V. Well-posedness of the kernel equations

In this section, we investigate the existence and uniqueness of the solution to system (IV.6) with boundary conditions (IV.8) as follows.

Theorem 5.1: Let M_1, M_2, M_3 and $\sigma_{ij}(x)(i,j = 1,2,3)$ be functions of $C^0[0,1]$. Suppose that (IV.9) holds. For the system (IV.6) with boundary conditions (IV.7)–(IV.8), there exists a unique L^∞ solution $K(x,y) = (k_{ij}(x,y))_{3\times 3}$ on the domain

$$\mathcal{T} = \{0 \le y \le x \le 1\}. \quad \text{(V.1)}$$

Moreover, we have

$$k_{ij}(x,y) \in C^0(\mathcal{T})(i \neq 1). \quad \text{(V.2)}$$

Proof: The kernel equations consist of three decoupled systems, more precisely, for each $i = 1,2,3$, the equations for k_{i1}, k_{i2} and k_{i3} are coupled together, and decoupled from the other kernels. Besides, consider the following change of variables

$$\chi = 1 - y, \qquad \eta = 1 - x \quad \text{(V.3)}$$

and

$$\hat{k}_{21}(\chi,\eta) = k_{21}(x,y) = k_{21}(1-\eta, 1-\chi),$$
$$\hat{k}_{22}(\chi,\eta) = k_{22}(x,y) = k_{22}(1-\eta, 1-\chi),$$
$$\hat{k}_{23}(\chi,\eta) = k_{23}(x,y) = k_{23}(1-\eta, 1-\chi).$$

Then, the system for $\hat{k}_{21}$, $\hat{k}_{22}$ and $\hat{k}_{23}$ rewrites

$$\lambda_1 \frac{\partial \hat{k}_{21}}{\partial \chi}(\chi,\eta) + \lambda_2 \frac{\partial \hat{k}_{21}}{\partial \eta}(\chi,\eta) = \sum_{j=1}^{3} \hat{k}_{2j}(\chi,\eta)\Sigma_{j1}(1-\chi) \quad \text{(V.4)}$$

$$\lambda_2 \frac{\partial \hat{k}_{22}}{\partial \chi}(\chi,\eta) + \lambda_2 \frac{\partial \hat{k}_{22}}{\partial \eta}(\chi,\eta) = \sum_{j=1}^{3} \hat{k}_{2j}(\chi,\eta)\Sigma_{j2}(1-\chi) \quad \text{(V.5)}$$

$$-\lambda_3 \frac{\partial \hat{k}_{23}}{\partial \chi}(\chi,\eta) + \lambda_2 \frac{\partial \hat{k}_{23}}{\partial \eta}(\chi,\eta) = \sum_{j=1}^{3} \hat{k}_{2j}(\chi,\eta)\Sigma_{j3}(1-\chi) \quad \text{(V.6)}$$

with boundary conditions

$$\begin{cases} \hat{k}_{21}(\chi,\chi) = \dfrac{\sigma_{21}(1-\chi)}{\lambda_2 - \lambda_1} \\ \hat{k}_{21}(\chi,0) = M_1(1-\chi) \\ \hat{k}_{22}(\chi,0) = M_2(1-\chi) \\ \hat{k}_{23}(\chi,\chi) = \dfrac{\sigma_{23}(1-\chi)}{\lambda_2 + \lambda_3} \end{cases} \quad \text{(V.7)}$$

and the domain for (χ,η) is also $\mathcal{T} = \{0 \le \eta \le \chi \le 1\}$. Then the well-posedness of $\hat{k}_{2j}(j = 1,2,3)$ guarantees the well-posedness of $k_{2j}(j = 1,2,3)$.

The following Lemma studies existence and uniqueness of a generic form of kernels that encompasses all three systems.

Lemma 5.1: Consider the following system

$$\epsilon_1 F^1_x(x,y) + \epsilon_1 F^1_y(x,y) = \sum_{j=1}^{4} c_{1j}(x,y)F^j(x,y)$$

$$\epsilon_2 F^2_x(x,y) + \epsilon_3 F^2_y(x,y) = \sum_{j=1}^{4} c_{2j}(x,y)F^j(x,y)$$

$$\mu_1 F^3_x(x,y) - \mu_2 F^3_y(x,y) = \sum_{j=1}^{4} c_{3j}(x,y)F^j(x,y)$$

$$\mu_1 F^4_x(x,y) - \mu_3 F^4_y(x,y) = \sum_{j=1}^{4} c_{4j}(x,y)F^j(x,y)$$

where $\forall i\ \epsilon_i, \mu_i > 0$, $c(x,y) \in C^0(\mathcal{T})$ and $\epsilon_2 > \epsilon_3$.

Consider also the following set of boundary conditions

$$F^1(x,0) = f_1(x) + p_1 F^3(x,0) \quad \text{(V.8)}$$

$$F^2(x,0) = f_2(x) + p_2 F^3(x,0) \quad \text{(V.9)}$$

$$F^2(x,x) = g_2(x) \quad \text{(V.10)}$$

$$F^3(x,x) = g_3(x) \quad \text{(V.11)}$$

$$F^4(x,x) = g_4(x) \quad \text{(V.12)}$$

where p_i are constants and f_i, $g_i \in \mathcal{C}^0([0,1])$. This system has a unique solution $F = (F_1,F_2,F_3,F_4)$ in $L^\infty(\mathcal{T})$. Moreover, if

$$g_2(0) = f_2(0) + p_2 g_3(0) \quad \text{(V.13)}$$

holds, we have that F_1, F_2 F_3 and F_4 are continuous on $\mathcal{T}$.

This Lemma is classically proved using the method of successive approximations, which is identical to the one in [5]. For this reason and brevity purposes, we will not detail it here.

VI. The inverse transformation and main result

The existence and unicity of the kernels K is simply obtained by applying Lemma 5.1 with, respectively

- For k_{11}, k_{12}, k_{13}, we suppose that

$$\begin{aligned} &F_1 = k_{11}, F_2 = k_{12}, F_3 = k_{13}, F_4 = 0, \\ &c_{4i}(x,y) \equiv 0 \quad (i = 1,2,3,4), \\ &\epsilon_1 = \epsilon_2 = \lambda_1, \epsilon_3 = \lambda_2, \\ &\mu_1 = \lambda_1, \mu_2 = \mu_3 = \lambda_3, \\ &f_1(x) = f_2(x) = g_4(x) \equiv 0, \\ &p_1 = \frac{\lambda_3}{\lambda_1}p, p_2 = \frac{\lambda_3}{\lambda_2}q, \\ &g_2(x) = \frac{\sigma_{12}(x)}{\lambda_1 - \lambda_2}, g_3(x) = \frac{\sigma_{13}(x)}{\lambda_1 + \lambda_3}. \end{aligned}$$

- For $\hat{k}_{21}$, $\hat{k}_{22}$, $\hat{k}_{23}$, noting the transformation (V.4)–(V.6), we suppose that

$$\begin{aligned} &F_1 = \hat{k}_{22}, F_2 = \hat{k}_{21}, F_3 = \hat{k}_{23}, F_4 = 0, \\ &c_{4i}(\chi,\eta) \equiv 0 \quad (i = 1,2,3,4), \\ &\epsilon_1 = \epsilon_3 = \lambda_2,\ \epsilon_2 = \lambda_1, \\ &\mu_1 = \lambda_3, \mu_2 = \mu_3 = \lambda_2, \\ &f_1(\chi) = M_2(1-\chi),\ p_1 = p_2 = g_4(\chi) \equiv 0, \\ &f_2(\chi) = M_1(1-\chi),\ g_2(\chi) = \frac{\sigma_{21}(1-\chi)}{\lambda_2 - \lambda_1}, \\ &g_3(\chi) = \frac{\sigma_{23}(1-\chi)}{\lambda_2 + \lambda_3}. \end{aligned}$$

- For k_{31}, k_{32}, k_{33}, we suppose that

$$\begin{aligned} &F_1 = k_{33}, F_2 = 0, F_3 = k_{31}, F_4 = k_{32}, \\ &c_{2i}(x,y) \equiv 0 \quad (i = 1,2,3,4), \\ &\epsilon_1 = \epsilon_2 = \epsilon_3 = \lambda_3, \\ &\mu_1 = \lambda_3, \mu_2 = \lambda_1, \mu_3 = \lambda_2, \\ &f_1(x) = M_3(x), p_1 = p_2 = f_2(x) \equiv 0, \\ &g_3(x) = -\frac{\sigma_{31}(x)}{\lambda_1 + \lambda_3}, g_4(x) = -\frac{\sigma_{32}(x)}{\lambda_2 + \lambda_3}. \end{aligned}$$

Noting that (IV.9) guarantees (V.13). ∎

Besides, transformation (IV.1) is a classical Volterra equation of the second kind, one can check from [10] and [16] that there exists a matrix function $\mathcal{R} \in (L^\infty(\mathcal{T}))^{3\times 3}$ such that

$$w(t,x) = \gamma(t,x) + \int_0^x \mathcal{R}(x,y)\gamma(t,y)dy. \quad \text{(VI.1)}$$

From the transformation (IV.1) evaluated at $x = 1$ and noting $\gamma(t,0) \equiv w(t,0)$, one gets the following feedback control laws

$$h_1(t) = \int_0^1 \sum_{j=1}^3 K_{1j}(1,x)w_j(t,x)dx \quad \text{(VI.2)}$$

$$h_2(t) = \int_0^1 \frac{1}{\lambda_2}\phi(1-x)w_2(t,x) + \int_0^1 \sum_{j=1}^3 \Big[K_{2j}(1,x) - \frac{1}{\lambda_2}\int_x^1 K_{2j}(s,x)\phi(1-s)ds\Big] w_j(t,x)dx \quad \text{(VI.3)}$$

Where $\phi(\cdot)$ is defined by

$$\phi(u) = g_{22}(u) - \int_0^u \frac{1}{\lambda_2} g_{22}(u-\nu)\phi(\nu)d\nu \quad \text{(VI.4)}$$

and satisfies $l(\tau,s) = \phi(\lambda_2\tau - \lambda_2 s)$, where $l(\cdot,\cdot)$ is defined by (III.5). This yields the following stabilization result.

Theorem 6.1: The mixed initial-boundary problem (II.1) with the boundary conditions (II.4), the feedback control law (VI.2)–(VI.3) and initial condition

$$t = 0 : w(0,x) = w_0(x), \quad \text{(VI.5)}$$

in which $w_0 \in (L^2(0,1))^3$, admits a $C^0([0,\infty);(L^2(0,1))^3)$ solutions $w = w(t,x)$, which vanishes in finite time $t \geq t_F$, where t_F is given by (III.8).

Proof. We obtain the explicit solutions of w from the direct and inverse transformation, as follows

$$w(t,x) = \gamma^*(t,x) + \int_0^x \mathcal{R}(x,y)\gamma^*(t,y)dy,$$

where $\gamma^*(t,x)$ is the solution of the system (III.1)-(III.3) with initial condition

$$\gamma_0(x) = w_0(x) - \int_0^x K(x,y)w_0(y)dy. \quad \text{(VI.6)}$$

From Proposition 3.1, we know that γ goes to zero in finite time $t = t_F$, therefore w also share this property. ∎

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