

Stabilization of coupled linear heterodirectional hyperbolic PDE–ODE systems

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Abstract

We solve the problem of stabilizing a linear ODE having a system of a linearly coupled hyperbolic PDEs in the actuating paths. The control design is based on a backstepping approach, yielding exponential closed-loop stability of the zero equilibrium.

Key words: Stabilization, Distributed Parameter Systems

1 Introduction

The interest for coupled Ordinary Differential Equations–Partial Differential Equations (ODE–PDE) systems has first emerged when considering delays in the actuating and sensing paths of ODE. Delays can be seen as first-order hyperbolic PDEs. There are many approaches to deal with input or measurement delays, usually divided into two categories: memoryless controllers, which extend standard control techniques without explicitly accounting for the delay in the control design [16,25,9]; and prediction-based controllers aiming at explicitly compensating the delay [20,4,2].

The use of Lyapunov and backstepping methods enabled dealing with more involved PDEs in the actuating and sensing paths. In [13], an output feedback control law is derived for an ODE having a heat equation in the actuating and sensing paths. The coupled PDE-ODE system is stabilized using an observer-controller structure relying on a backstepping approach. The same approach has been used to deal with ODEs coupled (rather than cascaded) with parabolic PDEs [21], uncertain parabolic PDEs [15], or ODE–Schrödinger cascades [17]. Lyapunov methods

enable the design of static output feedback controllers for nonlinear ODE–parabolic PDE cascades, as in [24].

The first application of the backstepping approach to deal with hyperbolic PDE–ODE couplings is [14] where actuator and sensor delays are explicitly compensated. While this problem had already been tackled by, e.g., the Smith predictor [20], the reformulation of the delay as a linear ODE enabled numerous related problem to be tackled, most notably non-constant and uncertain delays [2,4]. In [22], the problem of stabilizing a multi-input ODE with distinct delays is tackled using a backstepping approach. In [6], an observer is designed for an ODE having a homodirectional¹ hyperbolic PDE in the sensing path, relying on a Lyapunov analysis requiring to solve Linear Matrix Inequalities (LMI). As will appear, the systems of [22] are particular cases of the system considered here, although the control design approaches are different.

Here, we solve the problem of stabilizing an ODE with a system of first-order linear hyperbolic PDEs in the sensing

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¹ i.e. where all the states transport in the same direction

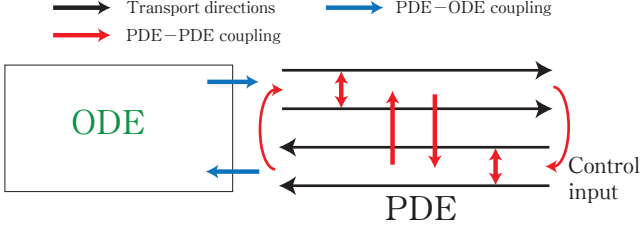


Fig. 1. Schematic view of the ODE-PDE coupling structure and actuating paths, i.e. we consider the following system

$$\dot{X}(t) = AX(t) + Bv(t, 0) \quad (1)$$

$$u_t(t, x) = -\Lambda^+ u_x(t, x) + \Sigma^{++} u(t, x) + \Sigma^{+-} v(t, x) \quad (2)$$

$$v_t(t, x) = \Lambda^- v_x(t, x) + \Sigma^{-+} u(t, x) + \Sigma^{--} v(t, x) \quad (3)$$

$$u(t, 0) = Q_0 v(t, 0) + CX(t) \quad (4)$$

$$v(t, 1) = R_1 u(t, 1) + U(t) \quad (5)$$

where $t > 0$ and $x \in [0, 1]$ are respectively the time and space variables, $X \in \mathbb{R}^p$ is the ODE state, $u(t, x) \in \mathbb{R}^n$ and $v(t, x) \in \mathbb{R}^m$ are the PDE states and $U(t)$ is the control input. The matrices Λ^+ and Λ^- are such that

$$\Lambda^+ = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \Lambda^- = \text{diag}(\mu_1, \dots, \mu_m) \quad (6)$$

with

$$-\mu_1 < \dots < -\mu_m < 0 < \lambda_1 < \dots < \lambda_n \quad (7)$$

The system naturally features several feedback loops or couplings that can be sources of instabilities:

- Inside the ODE itself (the A matrix in Equation (1))
- Coupling between hyperbolic states inside the spatial domain (the Σ^- matrices in (2),(3))
- Coupling between hyperbolic states at the boundary (the Q_0 and R_1 matrices in (4),(5))
- Coupling between the PDE and the ODE (the B and C matrices in (1),(4))
- A combination of all the above.

This structure is schematically depicted on Figure 1. This problem is motivated by applications in the drilling industry, more precisely the suppression of mechanical vibrations. Drilling systems are composed of long flexible strings subject to axial and torsional vibrations that propagate upwards and downwards. At the bottom end, the so-called *drill bit* crushes rock to create the borehole and is subject to friction and cutting forces. The ODE state X then corresponds to the drill bit axial and torsional positions while the PDE states represent the propagation of torsional and axial waves from and to the drill bit. For more details on these models, the interested reader is referred e.g. to [10,7].

When damping of the vibrations along the drillstring is neglected and the axial and torsional vibrations are coupled, the PDE reduces to *two* delay equations. Several contributions

have taken advantage of this simplification and designed stabilizing feedback laws, e.g. relying on neutral system approaches [19], flatness approaches [18] or predictor-based approaches [3,5]. However, no existing solution simultaneously allows stabilization

- taking into account damping inside the PDE domain (matrices $\Sigma^{\pm\pm}$ in (2),(3)) ;
- for a model of both axial *and* torsional vibrations, yielding 4 coupled PDE states rather than two delay equations [7,10].

Here, we solve these problems within the general setting of Equations (1)–(5). The system is mapped to an exponentially stable target system using a Volterra transformation. The target system has a cascade structure ensuring its convergence to the zero equilibrium. The design is based on a recent result on heterodirectional systems of hyperbolic PDEs [12].

The paper is organized as follows. In Section 2 we present the backstepping control design. In Section 3 we present a general well-posedness result for a class of hyperbolic PDEs on a triangular domain. In Section 4 we apply these results to the considered problem and state the main result. We conclude in Section 5 with perspectives for future work.

2 Control design

The control design is based on a Volterra transformation mapping the state (X, u, v) to a target system (X, α, β) with desirable properties. The target system equations are described in the next section.

2.1 Target system

We design the target system as follows

$$\dot{X}(t) = (A + B\mathcal{K})X(t) + B\beta(t, 0) \quad (8)$$

$$\begin{aligned} \alpha_t(t, x) = & -\Lambda^+ \alpha_x(t, x) + \Sigma^{++} \alpha(t, x) + \Sigma^{+-} \beta(t, x) + D(x)X(t) \\ & + \int_0^x C^+(x, y)\alpha(t, y)dy + \int_0^x C^-(x, y)\beta(t, y)dy \end{aligned} \quad (9)$$

$$\beta_t(t, x) = \Lambda^- \beta_x(t, x) + G(x)\beta(t, 0) + \bar{\Sigma}\beta(t, x) \quad (10)$$

$$\alpha(t, 0) = Q_0\beta(t, 0) + C_0X(t) \quad (11)$$

$$\beta(t, 1) = 0 \quad (12)$$

where C_0 , C^+ , C^- and D have yet to be defined and $G(\cdot)$ and $\bar{\Sigma}$ are defined as

$$G(x) = \begin{pmatrix} 0 & \dots & \dots & 0 \\ g_{2,1}(x) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ g_{m,1}(x) & \dots & g_{m,m-1} & 0 \end{pmatrix}, \quad \bar{\Sigma} = \text{diag}(\Sigma^{--}) \quad (13)$$

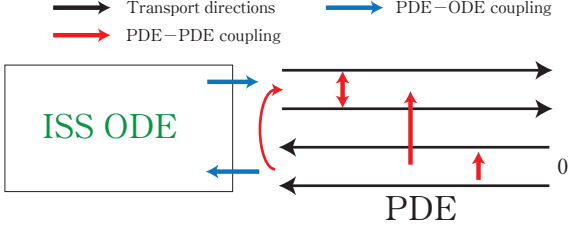


Fig. 2. Schematic view of the cascade structure of the target system. Here, ISS refers to Input-to-State Stability considering β as the (vanishing) input.

that is to say, $\bar{\Sigma}$ is the diagonal matrix consisting of the diagonal elements of Σ^- . The target system has a cascade structure schematically depicted on Figure 2. Its stability properties are assessed in the following Lemma.

Lemma 1 Denote \mathcal{T} the following closed triangular domain

$$\mathcal{T} = \{0 \leq y \leq x \leq 1\}. \quad (14)$$

Under the following assumptions

- (i) $A + B\mathcal{K}$ is Hurwitz
- (ii) $C^+, C^- \in L^\infty(\mathcal{T})$
- (iii) $G, D \in L^\infty([0, 1])$

the zero equilibrium of System (8)–(12) is exponentially stable in the L^2 sense.

Proof Consider the following Lyapunov functional

$$V(t) = \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} \alpha(t, x) dx + \int_0^1 e^{\delta x} \beta(t, x)^\top (\Lambda^-)^{-1} R \beta(t, x) dx + X(t)^\top P X(t) \quad (15)$$

where the symmetric definite positive matrix P , the diagonal matrix $R = \text{diag}(r_1, \dots, r_m)$ and the design parameter $\delta > 0$ are yet to be determined. Differentiating with respect to time

and integrating by parts yields

$$\begin{aligned} \dot{V}(t) = & \left[-e^{-\delta x} \alpha(t, x)^\top \alpha(t, x) + e^{\delta x} \beta(t, x)^\top R \beta(t, x) \right]_0^1 \\ & - \int_0^1 \delta e^{-\delta x} \alpha(t, x)^\top \alpha(t, x) dx - \int_0^1 \delta e^{\delta x} \beta(t, x)^\top R \beta(t, x) dx \\ & + 2 \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} [\Sigma^{++} \alpha(t, x) \\ & \quad + \Sigma^{+-} \beta(t, x) + D(x)X(t) \\ & \quad + \int_0^x C^+(x, y) \alpha(t, y) dy + \int_0^x C^-(x, y) \beta(t, y) dy] dx \\ & + 2 \int_0^1 e^{\delta x} \beta(t, x)^\top (\Lambda^-)^{-1} R [G(x) \beta(t, 0) + \bar{\Sigma} \beta(t, x)] dx \\ & + X(t)^\top [P(A + B\mathcal{K}) + (A + B\mathcal{K})^\top P] X(t) \\ & + X(t)^\top (PB + B^\top P) \beta(t, 0) \quad (16) \end{aligned}$$

Using Young's inequality and Assumptions (ii) and (iii) in Lemma 1 yields the existence of $M, \gamma > 0$ such that

$$\begin{aligned} 2 \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} \Sigma^{++} \alpha(t, x) dx \\ \leq M \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} \alpha(t, x) dx \quad (17) \end{aligned}$$

$$\begin{aligned} 2 \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} \Sigma^{+-} \beta(t, x) dx \\ \leq M \left(\int_0^1 e^{-\delta x} \alpha(t, x)^\top R^{-1} \alpha(t, x) dx \right. \\ \left. + \int_0^1 e^{\delta x} \beta(t, x)^\top R \beta(t, x) dx \right) \quad (18) \end{aligned}$$

$$\begin{aligned} 2 \int_0^1 \int_0^x e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} C^+(x, y) \alpha(t, y) dx dy \\ \leq \frac{M}{\delta} \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} \alpha(t, x) dx \quad (19) \end{aligned}$$

$$\begin{aligned} 2 \int_0^1 \int_0^x e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} C^-(x, y) \beta(t, y) dx dy \\ \leq \frac{M}{\delta} \left(\int_0^1 e^{-\delta x} \alpha(t, x)^\top R^{-1} \alpha(t, x) dx \right. \\ \left. + \int_0^1 e^{\delta x} \beta(t, x)^\top R \beta(t, x) dx \right) \quad (20) \end{aligned}$$

$$\begin{aligned} & \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} D(x) X(t) dx \\ & \leq M \left[\frac{1}{\gamma} |X(t)| + \gamma \int_0^1 e^{-\delta x} \alpha(t, x)^\top (\Lambda^+)^{-1} \alpha(t, x) dx \right] \end{aligned} \quad (21)$$

$$X(t)^\top (PB + B^\top P) \beta(t, 0) \leq M \left[\frac{1}{\gamma} |X(t)| + \gamma \beta(t, 0)^\top \beta(t, 0) \right] \quad (22)$$

where $\gamma > 0$ is a design parameter to be defined. Further, given the structure of the G matrix given by (13), there exists $M > 0$ such that

$$\begin{aligned} & \int_0^1 e^{\delta x} \beta(t, x)^\top (\Lambda^-)^{-1} R G(x) \beta(t, 0) dx \\ & \leq M \left[\int_0^1 e^{\delta x} \beta(t, x)^\top R \beta(t, x) dx + e^\delta \beta(t, 0)^\top C \beta(t, 0) \right] \end{aligned} \quad (23)$$

where

$$C = \text{diag}(c_1, \dots, c_m), \quad c_i = \begin{cases} \sum_{j=i+1}^m r_j, & 1 \leq i \leq m-1 \\ 0, & i = m \end{cases} \quad (24)$$

Besides, let $S = \text{diag}(s_1, \dots, s_m)$ such that

$$Q_0^\top Q_0 - S < 0 \quad (25)$$

Finally, plugging (8),(11),(12) into (16) and denoting $Q = -[P(A + BK) + (A + BK)^\top P] > 0$ yields

$$\begin{aligned} \dot{V}(t) & \leq -\beta(t, 0)^\top \left[R - MCe^\delta - \gamma M I_{m \times m} - S \right] \beta(t, 0) \\ & \quad - X(t)^\top \left[Q - \frac{2}{\gamma} I_{p \times p} \right] X(t) \\ & - \int_0^x e^{-\delta x} \alpha(t, x)^\top \left[(\delta - \gamma) I_{n \times n} - M \left(1 + \frac{1}{\delta} \right) (I_{n \times n} + R^{-1}) \right] \alpha(t, x) dx \\ & \quad - \left[\delta - M \left(2 + \frac{1}{\delta} \right) \right] \int_0^1 e^{\delta x} \beta(t, x)^\top R \beta(t, x) dx \end{aligned} \quad (26)$$

thus, picking γ such that

$$Q - \frac{2}{\gamma} I_{p \times p} < 0 \quad (27)$$

then $r_i, i = 1, \dots, m-1$ recursively such that

$$\forall i = 1, \dots, m-1 \quad r_i > M e^\delta \sum_{j=i+1}^m r_j + \gamma M + s_i \quad (28)$$

and r_m such that

$$r_m > \gamma M + s_m \quad (29)$$

and δ large enough concludes the proof.

2.2 Volterra Transform

To map the original system (1)–(5) to the target system (8)–(12), we use the following Volterra transformation

$$\alpha(t, x) \equiv u(t, x) \quad (30)$$

$$\begin{aligned} \beta(t, x) & = v(t, x) - \int_0^x K(x, y) u(t, y) dy \\ & \quad - \int_0^x L(x, y) v(t, y) dy - \gamma(x) X(t) \end{aligned} \quad (31)$$

where the kernels K, L and γ have yet to be defined. Differentiating (31) w.r.t. space and time yields the following kernel equations

for $1 \leq i \leq m, 1 \leq j \leq n$

$$\begin{aligned} \mu_i \partial_x K_{ij}(x, y) - \lambda_j \partial_y K_{ij}(x, y) & = -\sigma_{ii}^{--} K_{ij}(x, y) \\ & \quad + \sum_{k=1}^n \sigma_{kj}^{++} K_{ik}(x, y) + \sum_{k=1}^m \sigma_{kj}^{+-} L_{ik}(x, y) \end{aligned} \quad (32)$$

for $1 \leq i \leq m, 1 \leq j \leq m$

$$\begin{aligned} \mu_i \partial_x L_{ij}(x, y) + \mu_j \partial_y L_{ij}(x, y) & = -\sigma_{ii}^{--} L_{ij}(x, y) \\ & \quad + \sum_{k=1}^m \sigma_{kj}^{--} L_{ik}(x, y) + \sum_{k=1}^n \sigma_{kj}^{+-} K_{ik}(x, y) \end{aligned} \quad (33)$$

along with the following set of boundary conditions, for $i = 1, \dots, m$

$$\forall j, \quad K_{ij}(x, x) = -\frac{\sigma_{ij}^{--}}{\mu_i + \lambda_j} \quad (34)$$

$$\forall j \neq i, \quad L_{ij}(x, x) = -\frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \triangleq l_{ij} \quad (35)$$

$$\forall j \geq i \quad \mu_j L_{ij}(x, 0) = \sum_{k=1}^n \lambda_k K_{ik}(x, 0) q_{k,j} + \sum_{k=1}^p b_{kj} \gamma_{ik}(x) \quad (36)$$

where the $q_{k,j}$ in (36) are the elements of Q_0 . Besides, γ satisfies the following ODE

$$\begin{aligned} \forall j, \quad \mu_i \gamma'_{ij}(x) &= \sum_{k=1}^p \left[a_{kj} + \sum_{\substack{l=1 \\ l < i}}^m b_{kl} \kappa_{lj} \right] \gamma_{ik}(x) - \sigma_{ii}^- \gamma_{ij}(x) \\ &\quad - \sum_{\substack{k=1 \\ k < i}}^m \mu_k L^{ik}(x, 0) \kappa_{kj} \\ &\quad + \sum_{k=1}^n \lambda_k \left[c_{kj} + \sum_{\substack{l=1 \\ l < i}}^m q_{kl} \kappa_{lj} \right] K^{ik}(x, 0) \end{aligned} \quad (37)$$

with initial condition

$$\forall j, \quad \gamma_{ij}(0) = \kappa_{ij} \quad (38)$$

where the κ_{ij} are the entries of the control matrix gain \mathcal{K} . To ensure well-posedness of the system, we add the following arbitrary boundary conditions

$$\forall j < i, \quad L_{ij}(1, x) = l_{ij}(x) \quad (39)$$

These are degrees of freedom in the control design. However, their effect on the closed-loop performances are still unclear, thus, to study well-posedness, which we do in the next section, we only impose $l_{ij} \in L^\infty([0, 1])$. Besides, provided the K and L kernels are well-posed, the coefficients of G and the matrices C_0 , C^+ , C^- and D are given by

$$\begin{aligned} \forall j < i, \quad g_{ij}(x) &= \mu_j L_{ij}(x, 0) - \sum_{k=1}^n \lambda_k K_{ik}(x, 0) q_{k,j} \\ &\quad - \sum_{k=1}^p b_{kj} \gamma_{ik}(x) \end{aligned} \quad (40)$$

$$C^+(x, y) = \Sigma^{+-} K(x, y) + \int_y^x C^-(x, s) K(s, y) ds \quad (41)$$

$$C^-(x, y) = \Sigma^{+-} L(x, y) + \int_y^x C^-(x, s) L(s, y) ds \quad (42)$$

$$D(x) = \Sigma^{+-} \gamma(x) + \int_0^x C^-(x, y) \gamma(y) dy \quad (43)$$

$$C_0 = C + Q_0 \mathcal{K} \quad (44)$$

We prove well-posedness of the kernel equations over the next two sections. First, we study a relatively general class of hyperbolic PDEs on a triangular domain.

3 A general class of kernel equations

3.1 Problem setup

We consider the following class of equations on a triangular domain

$$\epsilon_i(x) \partial_x F_i(x, y) + \nu_i(y) \partial_y F_i(x, y) = \Sigma_i(x, y) F(x, y) \quad (45)$$

where $F = (F_1 \dots F_n)^T$. Each unknown F_i satisfies boundary conditions on a subset $\Omega_i \subset \partial\mathcal{T}$ of the following form

$$\forall i = 1, \dots, n \quad F_i|_{\Omega_i} = f_i + \sum_{j=1}^n \Gamma_{ij} F_j|_{\Omega_i} \quad (46)$$

where f_i and Γ_{ij} are functions of (x, y) defined on Ω_i . The functions Γ_{ij} , defined on the boundaries of the triangular domain \mathcal{T} , are boundary couplings between the different kernels F_i . The well-posedness of (45),(46) depends on the sparsity of the matrix $\Gamma = (\Gamma_{ij})$. More precisely, consider the following definition.

Definition 3.1 Let \mathcal{G} be the directed graph whose vertices are the F_i and whose edges are defined by the matrix $(\|\Gamma_{ij}\|_\infty)$. In other words, there is an edge between nodes i and j iff $\|\Gamma_{ij}\|_\infty \neq 0$. A valid path of length p in the graph is a p -uplet $a = (a_1, \dots, a_p)$ such that

$$\prod_{k=1}^{p-1} \|\Gamma_{a_k, a_{k+1}}\|_\infty \neq 0 \quad (47)$$

By convention, a path (a_1) of length $p = 1$ is the single node F_{a_1} .

The following Theorem gives a sufficient condition on the structure of \mathcal{G} for the system to be well-posed.

Theorem 3.2 Consider system (45) with boundary conditions (46). Assume

- (i) that the uncoupled system, obtained by taking $\Sigma(x, y) \equiv 0$ in (45) and $\Gamma_{ij} = 0, \forall i, j$ in (46), is well-posed;
- (ii) that there exists $\alpha > 1$ such that, for all $i = 1, \dots, n$, the following inequality holds

$$\forall (x, y) \in \mathcal{T} \quad \alpha \epsilon_i(x) - \nu_i(y) > \delta > 0 \quad (48)$$

- (iii) The graph \mathcal{G} is acyclic, i.e. it does not contain any cycles.

Then there is a unique solution $F \in L^\infty(\mathcal{T})$.

Remark 1 A necessary and sufficient condition for Assumption (i) to be satisfied is that, for every $i = 1, \dots, n$ the characteristics defined by the ϵ_i, ν_i uniquely connect each point of \mathcal{T} to Ω_i .

Remark 2 Assumption (ii) is a simple geometric condition for the well-posedness of the system: the tangent vector $(\epsilon_i(x), \mu_i(y))$ to all the characteristics, at all points $(x, y) \in \mathcal{T}$ must lie in the half-space such that the scalar product with $(\alpha, -1)^T$ is negative. In other words, the characteristics leaving the boundaries where (46) are defined must always “point away” from a certain line $y = \alpha x$, with $\alpha > 1$. Examples of such characteristics are pictured on Figure 3.

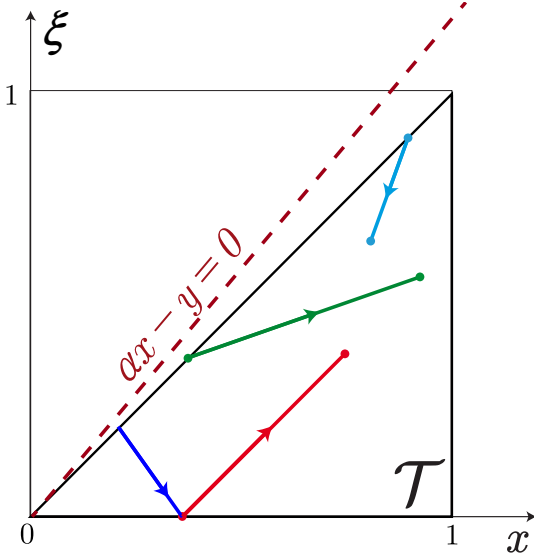


Fig. 3. Examples of characteristic lines that satisfy Assumption (ii) of Theorem 3.2.

The proof of Theorem 3.2 is quite involved and spans over the next few sections. It relies on the transformation of (45),(46) into integral equations. For this, we define in the next section the characteristic curves.

3.2 Transformation into integral equations

Assumption (i) of Theorem 3.2 yields the existence and uniqueness of characteristic curves, defined as follows

Definition 3.3 For each $i = 1, \dots, n$ and any $(x, y) \in \mathcal{T}$ there exists $(\chi_i^0(x, y), \xi_i^0(x, y)) \in \Omega_i$ and $s_i^F(x, y) \in \mathbb{R}^+$ such that

$$\begin{cases} \frac{d\chi_i(s; x, y)}{ds} = \epsilon_i(\chi_i(s; x, y)) \\ \chi_i(0; x, y) = \chi_i^0(x, y) \in \Omega_i \\ \chi_i(s_i^F(x, y); x, y) = x \end{cases} \quad (49)$$

$$\begin{cases} \frac{d\xi_i(s; x, y)}{ds} = \nu_i(\xi_i(s; x, y)) \\ \xi_i(0; x, y) = \xi_i^0(x, y) \in \Omega_i \\ \xi_i(s_i^F(x, y); x, y) = y \end{cases} \quad (50)$$

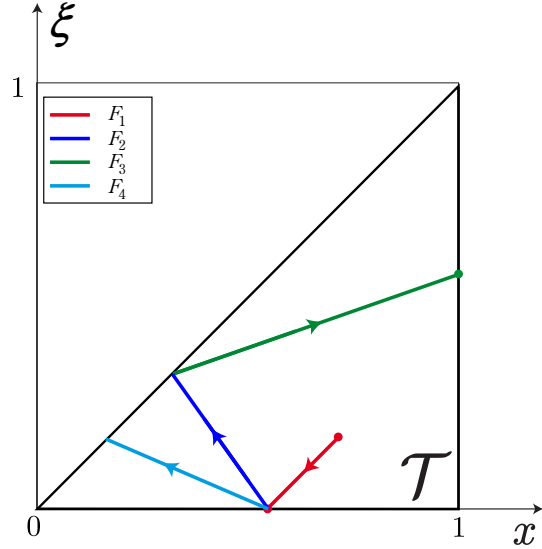
The curves $(\chi_i(s), \xi_i(s))$ are the characteristic curves associated with F_i . For any two points $(M_1, M_2) \in \mathcal{T}$, we denote $C_i(M_1, M_2)$ the characteristic curve associated with F_i starting in $M_1 = (x_1, y_1)$ and ending in $M_2 = (x_2, y_2)$, if such a curve exists, i.e. if

$$\chi_i(s^F(x_1, y_1); x_1, y_1) = x_2, \quad \xi_i(s^F(x_1, y_1); x_1, y_1) = y_2 \quad (51)$$

In the absence of the boundary couplings Γ_{ij} , the proof of well-posedness would consist in integrating (45) along (49),(50) and using a method of successive approximations. Here, this yields

$$\begin{aligned} F_i(x, y) &= f_i(\chi_i^0(x, y), \xi_i^0(x, y)) \\ &+ \sum_{j=1}^n \Gamma_{ij}(\chi_i^0(x, y), \xi_i^0(x, y)) F_j(\chi_i^0(x, y), \xi_i^0(x, y)) \\ &+ \int_0^{s_i^F(x, y)} \Sigma_i(\chi_i(s; x, y), \xi_i(s; x, y)) F(\chi_i(s; x, y), \xi_i(s; x, y)) ds \end{aligned} \quad (52)$$

The second term still contains unknowns, and the method of successive approximations does not straightforwardly apply. Rather, the second term must, again, be integrated along the characteristics of the F_j 's for which Γ_{ij} is non-zero. This situation is depicted on Figure 3.2 for an example.



To avoid this situation repeating infinitely (infinitely many “rebounds”), we use the following basic results from Graph Theory.

3.3 Basic results from Graph Theory

The following Definitions and Lemmas are classical results, see e.g. [23].

Lemma 2 If \mathcal{G} is acyclic, then the following holds

(a) There exist terminal nodes, i.e. there exists a set $TN \subset \{1, \dots, n\}$ such that

$$\forall i \in TN, \forall k \in \{1, \dots, n\} \quad \Gamma_{ik} = 0 \quad (53)$$

(b) All the valid paths are of finite length. In other words, any valid path can be completed with a valid path that leads to a terminal node and has a uniformly bounded length.

This allows us to add the following two definitions

Definition 3.4 For any node F_i , we define its depth d_i as the length of the longest valid path to a terminal node. We also define d^{\max} as the maximum length of any path $d^{\max} = \max_{i=1, \dots, n} d_i$.

Definition 3.5 Let $a = (a_1, \dots, a_p)$ be a (not necessarily valid) path. Then, we recursively define the sequence of points $M_k^a(x, y) \in \mathcal{T}$, $k = 0, \dots, p$ such that

$$M_0^a(x, y) = (x, y) \quad (54)$$

$$M_k^a(x, y) = \left(\chi_{a_k}^0(M_{k-1}^a), \xi_{a_k}^0(M_{k-1}^a) \right) \in \Omega_{a_k} \quad (55)$$

where $\chi_{a_k}^0(\cdot)$ and $\xi_{a_k}^0(\cdot)$ are defined by (49),(50). In other words, M_k^a is the point on the boundary of \mathcal{T} such that the characteristic curve $C_{a_k}(M_k^a, M_{k-1}^a)$ exists.

Property 3.6 For any two paths $a = (a_1, \dots, a_p)$ and $b = (b_1, \dots, b_q)$ one has

$$M_p^a(M_q^b(x, y)) = M_{p+q}^{(b,a)}(x, y) \quad (56)$$

where (b, a) denotes the concatenation of the two paths.

We are now ready to prove Theorem 3.2.

3.4 Proof of Theorem 3.2

Proof Classically, the proof consists in transforming the PDEs into integral equations and using a method of successive approximations.

3.4.1 Transformation into integral equations

The proof relies on the following transformation of (45),(46) into integral equations. For any $i = 1, \dots, n$, any $M = (x, y) \in \mathcal{T}$, one has

$$F_i(x, y) = \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \left(\prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}}(M_k^a(x, y)) \right) \cdot \left(f_{a_p}(M_p^a(x, y)) + \int_{C_{a_p}(M_p^a(x, y), M_{p-1}^a(x, y))} \Sigma_{a_p} F \right) \quad (57)$$

Remark 3 The sum $\sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}}$ denotes the sum over all (possibly invalid) paths of length p starting from the node F_i . However, a large number of the terms of this sum is zero due to the product of $\Gamma_{a_k, a_{k+1}}$ inside this sum.

We now prove Equation (57) by recursion on the depth d .

Case $d = 1$. Consider a node F_i such that $d_i = 1$, i.e. F_i is a terminal node. Assuming by convention that, the empty product is equal to 1, i.e.

$$\prod_{k=1}^0 \Gamma_{a_k, a_{k+1}} \equiv 1 \quad (58)$$

Equation (57) can be rewritten as

$$\begin{aligned} F_i(x, y) &= \sum_{p=1}^1 \sum_{\substack{a=(a_1) \\ a_1=i}} f_{a_1}(M_1^a(x, y)) + \int_{C_{a_1}(M_1^a(x, y), M_0^a(x, y))} \Sigma_{a_1} F \\ &= f_i(M_1^a(x, y)) + \int_{C_i(M_1^a(x, y), M_0^a(x, y))} \Sigma_i F \end{aligned} \quad (59)$$

Which exactly corresponds to integrating (45) along the characteristics associated to F_i since for all $j = 1, \dots, n$, $\Gamma_{ij} = 0$ for a terminal node (see (52) with $\Gamma_{ij} = 0$).

Recursion $d \rightarrow d + 1$. Assume now that (57) is true for all nodes of depth less or equal to d , for some $d \in \{1, \dots, d^{\max}\}$. Consider now F_i of depth $d + 1$. Integrating (45) along the characteristics and plugging in the boundary conditions (46) yields

$$\begin{aligned} F_i(x, y) &= f_i(M_1^{(i)}(x, y)) + \sum_{j=1}^n \Gamma_{ij}(M_1^{(i)}(x, y)) F_j(M_1^{(i)}(x, y)) \\ &\quad + \int_{C_i(M_1^{(i)}(x, y), M_0^{(i)}(x, y))} \Sigma_i F \end{aligned} \quad (60)$$

Notice that all the F_j for which $\Gamma_{ij} \neq 0$ are of depth $d_j \leq d$. Applying equation (57) to them yields

$$\begin{aligned} F_i(x, y) &= f_i(M_1^{(i)}(x, y)) + \int_{C_i(M_1^{(i)}(x, y), M_0^{(i)}(x, y))} \Sigma_i F \\ &\quad + \sum_{j=1}^n \sum_{p=1}^d \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=j}} \Gamma_{ij}(M_1^{(i)}(x, y)) \left(\prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}}(M_k^a(M_1^{(i)})) \right) \\ &\quad \cdot \left(f_{a_p}(M_p^a(M_1^{(i)})) + \int_{C_{a_p}(M_p^a(M_1^{(i)}), M_{p-1}^a(M_1^{(i)}))} \Sigma_{a_p} F \right) \end{aligned} \quad (61)$$

which yields

$$F_i(x, y) = f_i(M_1^{(i)}(x, y)) + \int_{C_i(M_1^{(i)}(x, y), M_0^{(i)}(x, y))} \Sigma_i F \\ + \sum_{p=1}^d \sum_{j=1}^n \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=j}} \left(\Gamma_{ij}(M_1^{(i)}(x, y)) \prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}} \left(M_k^a(M_1^{(i)}) \right) \right) \\ \cdot \left(f_{a_p}(M_p^a(M_1^{(i)})) + \int_{C_{a_p}(M_p^a(M_1^{(i)}), M_{p-1}^a(M_1^{(i)}))} \Sigma_{a_p} F \right) \quad (62)$$

i.e.

$$F_i(x, y) = f_i(M_1^{(i)}(x, y)) + \int_{C_i(M_1^{(i)}(x, y), M_0^{(i)}(x, y))} \Sigma_i F \\ + \sum_{p=1}^d \sum_{j=1}^n \sum_{\substack{a=(a_1, \dots, a_{p+1}) \\ a_2=j, a_1=i}} \left(\prod_{k=1}^p \Gamma_{a_k, a_{k+1}} \left(M_k^a(x, y) \right) \right) \\ \cdot \left(f_{a_{p+1}}(M_{p+1}^a(x, y)) + \int_{C_{a_{p+1}}(M_{p+1}^a(x, y), M_p^a(x, y))} \Sigma_{a_{p+1}} F \right) \quad (63)$$

i.e.

$$F_i(x, y) = f_i(M_1^{(i)}(x, y)) + \int_{C_i(M_1^{(i)}(x, y), M_0^{(i)}(x, y))} \Sigma_i F \\ + \sum_{p=1}^d \sum_{\substack{a=(a_1, \dots, a_{p+1}) \\ a_1=i}} \left(\prod_{k=1}^p \Gamma_{a_k, a_{k+1}} \left(M_{k+1}^a(x, y) \right) \right) \\ \cdot \left(f_{a_{p+1}}(M_{p+1}^a(x, y)) + \int_{C_{a_{p+1}}(M_{p+1}^a(x, y), M_p^a(x, y))} \Sigma_{a_{p+1}} F \right) \quad (64)$$

i.e.

$$F_i(x, y) = f_i(M_1^{(i)}(x, y)) + \int_{C_i(M_1^{(i)}(x, y), M_0^{(i)}(x, y))} \Sigma_i F \\ + \sum_{p=2}^{d+1} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \left(\prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}} \left(M_{k+1}^a(x, y) \right) \right) \\ \cdot \left(f_{a_p}(M_p^a(x, y)) + \int_{C_{a_p}(M_p^a(x, y), M_{p-1}^a(x, y))} \Sigma_{a_p} F \right) \quad (65)$$

i.e., using (58)

$$F_i(x, y) = \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \left(\prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}} \left(M_{k+1}^a(x, y) \right) \right) \\ \cdot \left(f_{a_p}(M_p^a(x, y)) + \int_{C_{a_p}(M_p^a(x, y), M_{p-1}^a(x, y))} \Sigma_{a_p} F \right) \quad (66)$$

which concludes the proof by induction since $d_i = d + 1$.

3.4.2 Method of successive approximations

The end of the proof follows the classical successive approximations method, applied to (57). More precisely, we define the following operators $\Phi = (\Phi_1 \dots \Phi_n)^\top$

$$\Phi_i[F](x, y) = \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \left(\prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}} \left(M_k^a(x, y) \right) \right) \int_{C_{a_p}(M_p^a(x, y), M_{p-1}^a(x, y))} \Sigma_{a_p} F \quad (67)$$

As well as the following vector $\phi = (\phi_1 \dots \phi_n)^\top$

$$\phi_i(x, y) = \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \left(\prod_{k=1}^{p-1} \Gamma_{a_k, a_{k+1}} \left(M_k^a(x, y) \right) \right) f_{a_p} \left(M_p^a(x, y) \right) \quad (68)$$

Define now the following sequence for $q \in \mathbb{N}$

$$\mathcal{F}^0(x, y) = 0 \quad (69)$$

$$\mathcal{F}^{q+1}(x, y) = \phi(x, y) + \Phi[\mathcal{F}^q](x, y) \quad (70)$$

$$= \left(\phi_1(x, y) \dots \phi_n(x, y) \right)^\top + \left(\Phi_1[\mathcal{F}^q](x, y) \dots \Phi_n[\mathcal{F}^q](x, y) \right)^\top \quad (71)$$

Finally, define the following sequence for $q \geq 1$

$$\Delta \mathcal{F}^q = \mathcal{F}^q - \mathcal{F}^{q-1} \quad (72)$$

Provided the limit exists, then

$$F = \lim_{q \rightarrow +\infty} \mathcal{F}^q = \sum_{q=1}^{+\infty} \Delta \mathcal{F}^q \quad (73)$$

is a solution to (57). To prove that the series is convergent, we rely on the following lemmas

Lemma 3 Assume inequality (48) holds. Then for all $i = 1, \dots, n$, $(x, y) \in \mathcal{T}$, the following function

$$\psi_{(x, y)}^i : s \in [0, s_i^F(x, y)] \mapsto \alpha \chi_i(s; x, y) - \xi_i(s; x, y) \quad (74)$$

is strictly increasing. In particular, the following inequality holds

$$\psi_{(x, y)}^i(s_i^F(x, y)) = \alpha x - y > \alpha \chi_i^0(x, y) - \xi_i^0(x, y) = \psi_{(x, y)}^i(0) \quad (75)$$

Thus, $\psi_{(x,y)}^i$ defines a diffeomorphism of $[0, s_i^F(x,y)]$ onto its image $[\alpha\chi_i^0(x,y) - \xi_i^0(x,y), \alpha x - y]$.

Proof The proof is trivial since for $i = 1, \dots, n$ and $(x,y) \in \mathcal{T}$, one has

$$\frac{d\psi_{(x,y)}^i}{ds}(s) = \alpha\epsilon_i(\chi_i(s; x, y)) - \nu_i(\xi_i(s; x, y)) \quad (76)$$

and recalling (48).

Corollary 3.7 For any path $a = (a_1, \dots, a_p)$ of length $p \geq 0$ and any $k = 0, \dots, p$, denote $M_k^a(x, y) = (x_k, y_k)$. Then one has

$$\alpha x_k - y_k \leq \alpha x - y \quad (77)$$

Proof We prove the result by induction. For $k = 0$, given the definition of the M_k^a (Equation (54)), one has

$$\alpha x_0 - y_0 = \alpha x - y \quad (78)$$

Assume now that (77) is satisfied for some $k = 1, \dots, p-1$, then, by definition of the $M_k^a(\cdot, \cdot)$, one has

$$\alpha x_{k+1} - y_{k+1} = \alpha\chi_{a_k}^0(x_k, y_k) - \xi_{a_k}^0(x_k, y_k) \quad (79)$$

Using Lemma 3 for $i = a_{k+1}$, this yields

$$\alpha x_{k+1} - y_{k+1} < \alpha x_k - y_k \quad (80)$$

Then, using the induction assumption, this concludes the proof.

Lemma 4 For any $i = 1, \dots, n$ and any $(x, y) \in \mathcal{T}$, one has

$$\int_0^{s_i^F(x,y)} [\alpha\chi_i(s; x, y) - \xi_i(s; x, y)]^q ds \leq \frac{1}{\delta} \frac{(\alpha x - y)^{q+1}}{q+1} \quad (81)$$

where α and δ are defined by (48).

Proof Consider the following change of variables

$$\tau = \psi_{(x,y)}^i(s) \quad (82)$$

where $\psi_{(x,y)}^i(\cdot)$ is defined by (74). It yields

$$\int_0^{s_i^F(x,y)} [\alpha\chi_i(s; x, y) - \xi_i(s; x, y)]^q ds = \int_{\alpha\chi_i^0(x,y) - \xi_i^0(x,y)}^{\alpha x - y} \frac{\tau^p d\tau}{\alpha\epsilon_i(\bar{\chi}_i(\tau; x, y)) - \nu_i(\bar{\xi}_i(\tau; x, y))} \quad (83)$$

where we have abusively denoted $\bar{\chi}_i(\tau; x, y) = \chi_i((\psi_{(x,y)}^i)^{-1}(\tau); x, y)$ (resp. $\bar{\xi}_i(\tau; x, y) = \xi_i((\psi_{(x,y)}^i)^{-1}(\tau); x, y)$). Using (48) this

yields

$$\int_0^{s_i^F(x,y)} [\alpha\chi_i(s; x, y) - \xi_i(s; x, y)]^q ds < \frac{1}{\delta} \frac{[\alpha x - y]^{q+1} - [\alpha\chi_i^0(x, y) - \xi_i^0(x, y)]^{q+1}}{q+1} \quad (84)$$

Since $(\chi_i^0(x, y), \xi_i^0(x, y)) \in \mathcal{T}$ and $\alpha > 1$, one has $\alpha\chi_i^0(x, y) - \xi_i^0(x, y) > 0$ which yields the result.

Lemma 5 For any path $a = (a_1, \dots, a_p)$ of length $p \geq 0$ and any $k = 1, \dots, p$, one has

$$\int_{C_{a_k}(M_k^a(x,y), M_{k-1}^a(x,y))} [\alpha\chi_{a_k}(\cdot) - \xi_{a_k}(\cdot)]^q \leq \frac{1}{\delta} \frac{(\alpha x - y)^{q+1}}{q+1} \quad (85)$$

Proof Denoting $M_k^a(x, y) = (x_k, y_k)$, one has from Lemma 4

$$\int_{C_{a_k}(M_k^a(x,y), M_{k-1}^a(x,y))} [\alpha\chi_{a_k}(\cdot) - \xi_{a_k}(\cdot)]^q \leq \frac{1}{\delta} \frac{[\alpha x_{k-1} - y_{k-1}]^{q+1}}{q+1} \quad (86)$$

Applying Corollary 3.7 yields the results.

Lemma 6 Define

$$\bar{\phi} = \max_{i=1, \dots, n} \|\phi_i(\cdot, \cdot)\|_{L^\infty(\mathcal{T})}, \quad \bar{\Gamma} = \max_{i,j \in \{1, 2, \dots, n\}} \|\Gamma_{ij}\|_\infty \quad (87)$$

$$\bar{\Sigma} = \max_{(x,y) \in \mathcal{T}, i=1, \dots, n} \|\Sigma_i(x, y)\| \quad (88)$$

$$M = \frac{d^{\max}}{\delta} \left(\prod_{k=0}^{d^{\max}} (n-k) \right) \bar{\Gamma}^{d^{\max}-1} \bar{\Sigma} \quad (89)$$

Assume that for some $q \geq 1$, one has, for all $(x, y) \in \mathcal{T}$

$$\forall i = 1, \dots, n \quad |\Delta \mathcal{F}_i^q(x, y)| \leq \bar{\phi} \frac{M^q (\alpha x - y)^q}{q!} \quad (90)$$

Then, one has

$$\forall i = 1, \dots, n \quad |\Delta \mathcal{F}_i^{q+1}(x, y)| \leq \bar{\phi} \frac{M^{q+1} (\alpha x - y)^{q+1}}{(q+1)!} \quad (91)$$

Assume that (90) holds for some fixed $q \geq 1$. Then, one has, for all $i = 1, \dots, n$

$$|\Delta \mathcal{F}_i^{q+1}(x, y)| \quad (92)$$

$$= |\Phi_i[\Delta \mathcal{F}^q](x, y)| \quad (93)$$

$$\leq \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \bar{\Gamma}^{p-1} \int_{C_{a_p}(M_p^a(x,y), M_{p-1}^a(x,y))} |\Sigma_{a_p} \Delta \mathcal{F}^q| \quad (94)$$

Using (90) yields

$$\leq \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \bar{\Gamma}^{p-1} \bar{\Sigma} \int_{C_{a_p}(M_p^a(x,y), M_{p-1}^a(x,y))} \bar{\phi} \frac{M^q (\alpha \chi_{a_p}(\cdot) - \xi_{a_p}(\cdot))^q}{q!} \quad (95)$$

Using Lemma 5 yields

$$\leq \sum_{p=1}^{d_i} \sum_{\substack{a=(a_1, \dots, a_p) \\ a_1=i}} \bar{\Gamma}^{p-1} \bar{\Sigma} \bar{\phi} \frac{M^q (\alpha x - y)^{q+1}}{(q+1)!} \quad (96)$$

Noticing that there cannot be more than $\prod_{k=1}^p (n-k)$ paths of length p from a given node i , this yields

$$\leq d_i^{\max} \left(\prod_{k=1}^{d_i^{\max}} (n-k) \right) \bar{\Gamma}^{d_i^{\max}-1} \bar{\Sigma} \bar{\phi} \frac{M^q (\alpha x - y)^{q+1}}{(q+1)!} \quad (97)$$

which, in turn, yields the result given the definition of M (Equation (89)). Finally, Lemma 6 ensures that the series (72) is uniformly convergent, thus the kernel equations (45) with boundary conditions (46) are well-posed (see, e.g. [8] for a detailed proof). In the next section, we apply Theorem 3.2 to prove well-posedness of (32)–(39).

4 Well-posedness of (32)–(39) and control law

In this section, we apply the results of Section 3 to prove the well-posedness of the kernel equations. The following theorem assesses the well-posedness of the kernel equations.

Theorem 4.1 *System (32)–(39) has a unique solution $K, L \in L^\infty(\mathcal{T})$.*

Proof We prove the result by induction on $i = 1, \dots, n$.

$\mathbf{i} = 1$. For $i = 1$, the equations rewrite as follows

for $1 \leq j \leq n$

$$\mu_1 \partial_x K_{1j}(x, y) - \lambda_j \partial_\xi K_{1j}(x, y) = -\sigma_{11}^- K_{1j}(x, y) + \sum_{k=1}^n \sigma_{kj}^{++} K_{1k}(x, y) + \sum_{p=1}^m \sigma_{pj}^- L_{1p}(x, y) \quad (98)$$

for $1 \leq j \leq m$

$$\mu_1 \partial_x L_{1j}(x, y) + \mu_j \partial_\xi L_{1j}(x, y) = -\sigma_{11}^- L_{1j}(x, y) + \sum_{p=1}^m \sigma_{pj}^- L_{1p}(x, y) + \sum_{k=1}^n \sigma_{kj}^+ K_{1k}(x, y) \quad (99)$$

$$\forall j, \quad K_{1j}(x, x) = -\frac{\sigma_{1j}^{++}}{\mu_1 + \lambda_j} \triangleq k_{1j} \quad (100)$$

$$\forall j \neq 1, \quad L_{1j}(x, x) = -\frac{\sigma_{1j}^-}{\mu_1 - \mu_j} \triangleq l_{1j} \quad (101)$$

$$\forall j \geq 1 \quad \mu_j L_{1j}(x, 0) = \sum_{k=1}^n \lambda_k K_{1k}(x, 0) q_{k,j} + \sum_{k=1}^p b_{kj} \gamma_{1k}(x) \quad (102)$$

$$\forall j, \quad \mu_1 \gamma'_{1j}(x) = \sum_{k=1}^p a_{kj} \gamma_{1k}(x) - \sigma_{11}^- \gamma_{1j}(x) \quad (103)$$

$$+ \sum_{k=1}^n \lambda_k c_{kj} K^{1k}(x, 0) \quad (104)$$

$$\forall j, \quad \gamma_{1j}(0) = \kappa_{1j} \quad (105)$$

One can readily check that (98)–(99) are of the form (45). Besides, the ODE (104) can also be put under the form (45) by “embedding” it into \mathcal{T} . More precisely, denoting

$$\mathbb{1}_{\{y=0\}}(x, y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{otherwise} \end{cases} \quad (106)$$

one can define $\tilde{\gamma}_j$ such that

$$\forall (x, y) \in \mathcal{T} \quad \tilde{\gamma}_j(x, y) = \mathbb{1}_{\{y=0\}}(x, y) \gamma_{1j}(x) \quad (107)$$

or, equivalently, the $\tilde{\gamma}_j$ satisfy the following PDEs of the form (45)

$$\mu_1 \partial_x \tilde{\gamma}_j(x, y) = \mathbb{1}_{\{y=0\}}(x, y) \left[\sum_{k=1}^n a_{kj} \tilde{\gamma}_k(x, y) - \sigma_{11}^- \tilde{\gamma}_j(x, y) + \sum_{k=1}^n K^{1k}(x, y) \lambda_k c_{kj} \right] \quad (108)$$

with boundary conditions

$$\tilde{\gamma}_j(x, x) = \mathbb{1}_{\{y=0\}}(x, x) \kappa_{1j} \quad (109)$$

Besides, boundary conditions (100)–(102), (109) are of the form (46), with the boundary coupling coefficients Γ_{ij} being zero for every kernel except the L_{ij} on the $y = 0$ boundary. Therefore, the graph defined by Γ_{ij} is acyclic, and Theorem 3.2 applies to (98)–(105) which is well-posed, i.e. has a unique solution with $K_{1j}, L_{1j} \in L^\infty(\mathcal{T})$ and $\gamma_{1j} \in L^\infty([0, 1])$.

$\{\mathbf{1}, \dots, \mathbf{i} - \mathbf{1}\} \rightarrow \mathbf{i}$. Let $i \in \{2, \dots, m\}$ be fixed and assume that for $k = 1, \dots, i - 1$ there exist $K_{kj}, L_{kj} \in L^\infty(\mathcal{T})$ and $\gamma_{kj} \in L^\infty([0, 1])$, for all j . Then, Equations (32)–(39) are of the form (45), (46) with coefficients in L^∞ since they are linear in the K_{ij}, L_{ij} and γ_{ij} variables with coefficients that depend on the K_{kj}, L_{kj} and γ_{kj} for $k < j$. Thus, Theorem 3.2 applies again and the equations are well-posed.

This yields the main result of the paper, stated in the following theorem.

Theorem 4.2 Consider System (1)–(5) with the following control law

$$U(t) = -R_1 u(t, 1) + \int_0^1 K(1, y) u(t, y) dy + \int_0^1 L(1, y) v(t, y) dy + \gamma(1) X(t) \quad (110)$$

where K , L and γ are defined by (32)–(39). Then, the zero equilibrium is exponentially stable in the L^2 sense.

Proof Theorem 4.1 ensures the existence of $K, L \in L^\infty(\mathcal{T})$, $\gamma \in L^\infty([0, 1])$ such that (30),(31) holds and (α, β) satisfies (8),(12). Lemma 1 and the invertibility of the Volterra transformation yields the result.

5 Conclusion and perspectives

We have presented a control design for ODEs with a system of hyperbolic PDEs in the actuating path. The design results in a full-state feedback law needing measurements of the distributed actuator states along the spatial domain. This is not realistic in practice and future contributions will focus on the design of an observer solely relying on (collocated) boundary measurements.

Besides, the result opens the door to control design for other systems involving cascaded hyperbolic PDEs. In particular, networks of systems of hyperbolic balance laws are instrumental in modeling, e.g. oil production systems, networks of open channels [1] or power transmission lines [11].

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