

A backstepping boundary observer for a class of linear first-order hyperbolic systems

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Abstract— We investigate the boundary observer design problem for a class of linear first-order hyperbolic systems on a finite space domain with spatially varying parameters. The system features one negative transport speed and an arbitrary fixed number n of positive transport speeds. Using a backstepping approach, the distributed states are estimated from a single boundary measurement, as illustrated in presented numerical simulations.

I. INTRODUCTION

First-order linear hyperbolic systems are predominant in modelling of open channel flow [5], heat exchangers [17] or multiphase flow [14]. The coupling between states having opposite transport speed can make the zero equilibrium unstable, leading to undesirable behavior. The models can be used to design control laws with a view to improving the efficiency of water management systems [1] or processes such as oil recovery [9] or oil well drilling [13]. An important restriction to the control design lies in the absence, for the large majority of these industrial systems, of distributed measurements.

Stabilizing systems of first-order hyperbolic Partial Differential Equations (PDEs) on a finite space domain using boundary sensors only is a difficult task, since the instability often arises from in-domain terms that need to be cancelled. In many contributions [3], [5], [10], [12], controllers are sought under the form of static output feedback laws, i.e., the control law is a linear function of the measured outputs at the boundary, with carefully selected gains. These designs often lead to a set of sufficient conditions for stability, both on the in-domain coupling coefficients and the feedback gains, as in [1]. A similar result is derived for a first-order hyperbolic PDE system coupled with a linear system of Ordinary Differential Equations (ODE) at the boundary in [4].

Another approach consists in deriving full-state feedback laws, as is done in [16], [9] using the backstepping approach. Their main advantage is to stabilize the equilibrium in more general cases, i.e. with little more than boundedness and smoothness conditions on the coefficients. In this paper, we consider systems featuring an arbitrary fixed number n of states with strictly positive transport speeds (i.e. travelling left to right) and one state with a negative transport speed

(i.e. travelling right to left), controlled at the right boundary and measured at the left boundary. In [8], a full-state feedback law has been designed to exponentially stabilize the zero equilibrium of this class of systems. However, to be implementable, the controller requires the knowledge of the distributed states of the systems, which are usually not available. Thus, we propose in this paper an observer scheme to estimate the distributed states from a single boundary measurement.

Our approach is as follows. We construct an observer that is of Luenberger-type inside the domain (i.e. a copy of the system plus output error tracking terms), but has output injection terms at the left boundary. Using the backstepping approach, we stabilize the zero equilibrium of the estimation error dynamics. We design an invertible Volterra integral transformation that maps these to a target system with desirable stability properties. The transformation kernels are shown to satisfy a set of linear first-order hyperbolic PDEs on a triangular domain, of the same form as the one satisfied by the controller transformation kernels derived in [8]. Numerical simulations highlight the merits of the design, as well as the issues related to using output injection terms at the boundary.

The paper is organized as follows. In Section II we detail the system equations and formulate the estimation problem. In III we design the observer gains using the backstepping method. In Section IV we illustrate our approach with numerical simulations. Conclusions and open questions are presented in Section V.

II. PROBLEM DESCRIPTION

We consider the following set of linear PDEs on the spatial domain $x \in [0, 1]$

$$u_t^i(t, x) + \lambda_i(x)u_x^i(t, x) = \sum_{j=1}^n \sigma_{i,j}(x)u^j(t, x) + \omega_i(x)v(t, x), \quad (1)$$

$$v_x(t, x) - \mu(x)v_x(t, x) = \sum_{j=1}^n \theta_j(x)u^j \quad (2)$$

along with the following boundary conditions

$$u^i(t, 0) = q_i v(t, 0), \quad (3)$$

$$v(t, 1) = \sum_{j=1}^n \rho_j u^j(t, 1) + U(t), \quad (4)$$

$$v(t, 0) = y(t). \quad (5)$$

where the u^i , $i = 1, \dots, n$ and v are the distributed states, $U(t)$ is the control input and $y(t)$ the measured output. The

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transport velocities are assumed to satisfy the following inequalities

$$\forall x \in [0, 1] \quad -\mu(x) < 0 < \lambda_1(x) < \dots < \lambda_n(x) \quad (6)$$

which indicates that the u^i states evolve left to right, whereas the v state evolves right to left. Inequalities (6) guarantee the hyperbolicity of System (1)–(5) and the well-posedness of the mixed initial-boundary value problem (see, e.g., [11]). This setup is schematically depicted on Figure 1.

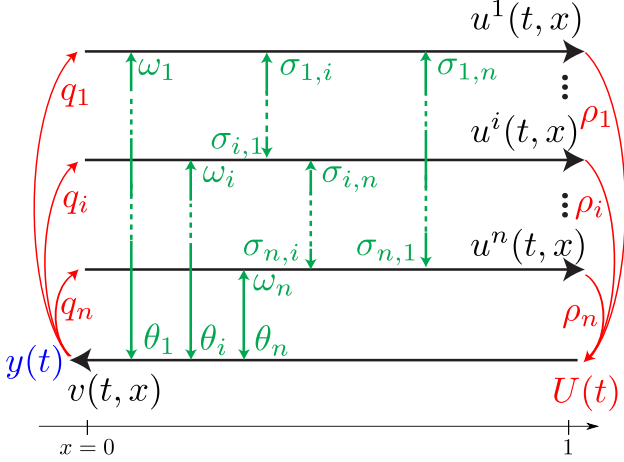


Fig. 1. Schematic view of the $(n+1)$ -state hyperbolic system. The space dependence of the coupling coefficients has been omitted for the sake of clarity.

Remark 1: In [8], the boundary stabilization problem is investigated for (1)–(3) but with a different condition at the right boundary, of the form $v(t, 1) = U(t)$. This is equivalent to Equation (4) in the case where the $u^j(t, 1)$, $j = 1, \dots, n$ are measured, which is not the case here. Thus, the right boundary condition considered here is slightly more general than the one in [8]. More importantly, it is more realistic as, if the system is derived from a nonlinear model, the boundary conditions are much likelier to be of the form (4) than the simpler form $v(t, 1) = U(t)$.

We consider the following observer, for $i = 1, \dots, n$

$$\begin{aligned} \partial_t \hat{u}_i(t, x) + \lambda_i(x) \partial_x \hat{u}_i(t, x) &= \sum_{j=1}^n \sigma_{i,j}(x) \hat{u}_j(t, x) + \omega_j(x) \hat{v}(t, x) \\ &\quad - p_i(x) [v(t, 0) - \hat{v}(t, 0)] \end{aligned} \quad (7)$$

$$\begin{aligned} \partial_t \hat{v}(t, x) - \mu(x) \partial_x \hat{v}(t, x) &= \sum_{j=1}^n \theta_j(x) \hat{u}_j(t, x) \\ &\quad - p_{n+1}(x) [v(t, 0) - \hat{v}(t, 0)] \end{aligned} \quad (8)$$

along with the following boundary conditions

$$\hat{u}_i(t, 0) = q_i v(t, 0), \quad \hat{v}(t, 1) = \sum_{j=1}^n \rho_j \hat{u}_j(t, 1) + U(t) \quad (9)$$

where the gains $p_i(\cdot)$, $i = 1, \dots, n+1$ are yet to be determined. In (9), one should notice the output injection term at the left boundary. As is illustrated in Section IV, these have undesirable properties as potential noise affecting the measurements

is not filtered by the observer. However, we could not design an observer without output injection at the left boundary. In the next section, we design the observer gains by mapping the error dynamics to an exponentially stable system using a Volterra transformation of the second kind.

III. BACKSTEPPING DESIGN

A. Target system

The observer error equations read

$$\begin{aligned} \tilde{u}_i^j(t, x) + \lambda_i(x) \tilde{u}_x^j(t, x) &= \sum_{j=1}^n \sigma_{i,j}(x) \tilde{u}^j(t, x) + \omega_i(x) \tilde{v}(t, x) \\ &\quad + p_i(x) \tilde{v}(t, 0), \end{aligned} \quad (10)$$

$$\tilde{v}_t(t, x) - \mu(x) \tilde{v}_x(t, x) = \sum_{j=1}^n \theta_j(x) \tilde{u}_j(t, x) + p_{n+1}(x) \tilde{v}(t, 0) \quad (11)$$

with boundary conditions

$$\tilde{u}^i(t, 0) = 0, \quad \tilde{v}(t, 1) = \sum_{j=1}^n \rho_j \tilde{u}^j(t, 1) \quad (12)$$

To stabilize the zero equilibrium of the error dynamics, we try to map (10)–(12) to the following target system

$$\tilde{\alpha}_t^j(t, x) + \lambda_j(x) \tilde{\alpha}_x^j(t, x) = \sum_{j=1}^n \sigma_{i,j}(x) \tilde{\alpha}^j(t, x) \quad (13)$$

$$+ \sum_{j=1}^n \int_0^x g_{i,j}(x, \xi) \tilde{\alpha}^j(t, \xi) d\xi, \quad (14)$$

$$\tilde{\beta}_t(t, x) - \mu(x) \tilde{\beta}_x(t, x) = \sum_{j=1}^n \theta_j(x) \tilde{\alpha}^j(t, x) \quad (15)$$

$$+ \sum_{j=1}^n \int_0^x h_j(x, \xi) \tilde{\alpha}^j(t, \xi) d\xi \quad (16)$$

with boundary conditions

$$\tilde{\alpha}^j(t, 0) = 0, \quad \tilde{\beta}(t, 1) = \sum_{j=1}^n \rho_j \tilde{\alpha}^j(t, 1) \quad (17)$$

where the $g_{i,j}(\cdot)$, $h_j(\cdot)$ are functions to be determined on the triangular domain

$$\mathcal{T} = \{ (x, \xi) \in \mathbb{R}^2 \mid 0 \leq \xi \leq x \leq 1 \}. \quad (18)$$

The stability properties of this system, which is schematically depicted on Figure 2, are stated in the following lemma

Lemma 3.1: Under the following assumptions

$$\begin{aligned} \forall i, j = 1, \dots, n \quad \lambda_i, \mu &\in C^1([0, 1], \mathbb{R}_+^*), \quad \sigma_{i,j}, \theta_i \in \mathcal{L}^\infty([0, 1]) \\ \alpha_0^i, \beta_0 &\in \mathcal{L}^2([0, 1]), \quad g_{i,j}, h_i \in \mathcal{L}^\infty(\mathcal{T}) \end{aligned}$$

the equilibrium $(\tilde{\alpha}^1, \dots, \tilde{\alpha}^n, \tilde{\beta})^T \equiv (0, \dots, 0, 0)^T$ of system (14),(16) with boundary conditions (17) and initial conditions $(\alpha_0^1, \dots, \alpha_0^n, \beta_0)^T$ is exponentially stable in the \mathcal{L}^2 sense.

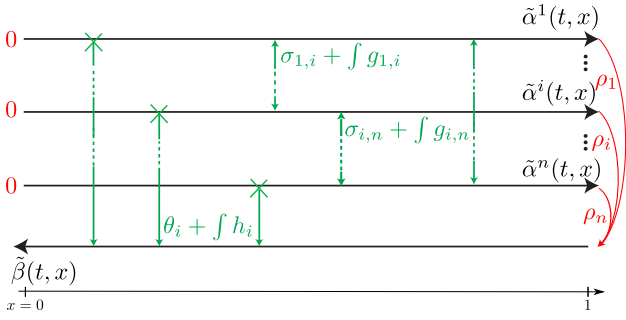


Fig. 2. Schematic representation of the observer error target system.

Proof: Stability can be shown using the following Lyapunov functional

$$V(t) = \int_0^1 p e^{-\delta x} \sum_{i=1}^n \frac{\tilde{\alpha}_i(t, x)^2}{\lambda_i(x)} dx + \int_0^1 \frac{1+x}{\mu(x)} \tilde{\beta}(t, x)^2 dx \quad (19)$$

with carefully selected design parameters $p > 0$ and $\delta > 0$, similarly to the Proof of Lemma 1 in [8]. A more intuitive approach consists in recognizing that, in (14)–(17), the $(\tilde{\alpha}_1, \dots, \tilde{\alpha}_n)$ sub-system (which is exponentially stable since it consists of homo-directional states only¹) drives the $\tilde{\beta}$ sub-system, itself exponentially stable for $\tilde{\alpha}_1 = \dots = \tilde{\alpha}_n = 0$. Thus, the cascade system is exponentially stable. ■

B. Backstepping transformation

To map System (10)–(12) to (14)–(17), we consider a backstepping transformation of the form

$$\forall i = 1, \dots, n \quad \tilde{u}^i(t, x) = \tilde{\alpha}^i(t, x) + \int_0^x m^i(x, \xi) \tilde{\beta}(t, \xi) d\xi, \quad (20)$$

$$\tilde{v}(t, x) = \tilde{\beta}(t, x) + \int_0^x m^{n+1}(x, \xi) \tilde{\beta}(t, \xi) d\xi \quad (21)$$

where the kernels $m^i(\cdot)$, $i = 1, \dots, n+1$ are defined on the triangular domain \mathcal{T} . We now look for sufficient conditions on the kernels. Differentiating (20) with respect to time and space, using (14)–(17) and plugging the result in (10) yields, after some computation, for each $i = 1, \dots, n$

$$\begin{aligned} 0 = & \left[\mu(x) m^i(x, x) + \lambda_i(x) m^i(x, x) - \omega_i(x) \right] \tilde{\beta}(x) \\ & - \left[\mu(0) m^i(x, 0) + p_i(x) \right] \tilde{\beta}(0) + \sum_{j=1}^n \int_0^x \left[g_{i,j}(x, \xi) \right. \\ & \left. + \theta_j(\xi) m^i(x, \xi) + \int_{\xi}^x m^i(x, s) h_j(s, \xi) ds \right] \alpha^j(\xi) d\xi \\ & + \int_0^x \left[\lambda_i(x) m_x^i(x, \xi) - \mu(\xi) m_{\xi}^i(x, \xi) - \mu'(\xi) m^i(x, \xi) \right. \\ & \left. - \omega_i(x) m^{n+1}(x, \xi) - \sum_{j=1}^n \sigma_{i,j}(x) m^j(x, \xi) \right] \tilde{\beta}(\xi) d\xi \quad (22) \end{aligned}$$

¹i.e., states evolving in the same direction.

Similarly, differentiating (21) with respect to time and space, using (14)–(17) and plugging the result in (11) yields

$$\begin{aligned} 0 = & - \left[\mu(0) m^{n+1}(x, 0) + p_{n+1}(x) \right] \tilde{\beta}(0) + \sum_{i=1}^n \int_0^x \left[h_i(x, \xi) \right. \\ & \left. + \theta_i(\xi) m^{n+1}(x, \xi) + \int_{\xi}^x n(x, s) h_j(s, \xi) ds \right] \tilde{\alpha}^j(\xi) d\xi \\ & - \int_0^x \left[\mu(x) m_x^{n+1}(x, \xi) + \mu(\xi) m_{\xi}^{n+1}(x, \xi) + \mu'(\xi) m^{n+1}(x, \xi) \right. \\ & \left. + \sum_{j=1}^n \theta_j(x) m^j(x, \xi) \right] \tilde{\beta}(\xi) d\xi \quad (23) \end{aligned}$$

Finally, taking $x = 1$ in (21) and plugging in (12),(17) yields

$$0 = \int_0^1 \left[m^{n+1}(1, \xi) - \sum_{j=1}^n \rho_j m^j(1, \xi) \right] \tilde{\beta}(\xi) d\xi \quad (24)$$

Thus, a sufficient condition for the transformation (20),(21) to map the original system to the target system is that the kernels m^i satisfy the following system of hyperbolic PDEs on the triangular domain \mathcal{T}

$$\left\{ \begin{aligned} \lambda_1(x) m_x^1 - \mu(\xi) m_{\xi}^1 &= \mu'(\xi) m^1 + \sum_{j=1}^n \sigma_{1,j}(x) m^j + \omega_1(x) m^{n+1} \\ &\vdots \\ \lambda_i(x) m_x^i - \mu(\xi) m_{\xi}^i &= \mu'(\xi) m^i + \sum_{j=1}^n \sigma_{i,j}(x) m^j + \omega_i(x) m^{n+1} \\ &\vdots \\ \lambda_n(x) m_x^n - \mu(\xi) m_{\xi}^n &= \mu'(\xi) m^n + \sum_{j=1}^n \sigma_{n,j}(x) m^j + \omega_n(x) m^{n+1} \\ \mu(x) m_x^{n+1} + \mu(\xi) m_{\xi}^{n+1} &= -\mu'(\xi) m^{n+1} - \sum_{i=1}^n \theta_i(x) m^i \end{aligned} \right. \quad (25)$$

with boundary conditions

$$\left\{ \begin{aligned} m^1(x, x) &= \frac{\omega_1(x)}{\lambda_1(x) + \mu(x)} \\ &\vdots \\ m^i(x, x) &= \frac{\omega_j(x)}{\lambda_i(x) + \mu(x)} \\ &\vdots \\ m^n(x, x) &= \frac{\omega_n(x)}{\lambda_n(x) + \mu(x)} \\ m^{n+1}(1, \xi) &= \sum_{j=1}^n \rho_j m^j(1, \xi) \end{aligned} \right. \quad (26)$$

Besides, the observer gains are given by

$$\forall i = 1, \dots, n+1 \quad p_i(x) = -\mu(0) m^i(x, 0). \quad (27)$$

The existence, uniqueness and smoothness of solutions to (25),(26) directly follows from the corresponding result in the control design case detailed in [8]. More precisely,

considering Theorem 5.3 in [8], the following Corollary holds

Corollary 3.2: Under the following assumptions

$$\forall i, j = 1, \dots, n \quad \lambda_i, \mu \in C^1([0, 1], \mathbb{R}^*), \quad \sigma_{i,j}, \omega_i, \theta_i \in \mathcal{L}^\infty([0, 1]) \\ \mathbf{u}^0, \mathbf{v}^0 \in \mathcal{L}^2([0, 1])$$

System (25) with boundary conditions (26) admits a unique continuous solution $(m^1(\cdot) \dots m^{n+1}(\cdot))$ on \mathcal{T} .

Proof: The result is directly obtained by performing the following variable change

$$\hat{\xi} = 1 - x, \quad \hat{x} = 1 - \xi, \quad (28)$$

and applying Theorem 5.3 in [8] to the resulting system of hyperbolic PDEs on \mathcal{T} . ■

C. Inverse transformation

Transformation (21) is a scalar Volterra integral equation of the second type. Since m^{n+1} is continuous by Corollary 3.2, there exists a unique continuous inverse kernel $r^{n+1}(\cdot)$ (see, e.g., [15]) such that

$$\tilde{\beta}(t, x) = \tilde{v}(t, x) + \int_0^x r^{n+1}(x, \xi) \tilde{v}(t, \xi) d\xi \quad (29)$$

implicitly defined on \mathcal{T} by

$$r^{n+1}(x, \xi) = m^{n+1}(x, \xi) + \int_\xi^x m^{n+1}(x, s) r^{n+1}(s, \xi) ds. \quad (30)$$

Besides, plugging (29) into (20) yields, for each $i = 1, \dots, n$

$$\begin{aligned} \tilde{\alpha}^i(t, x) &= \tilde{u}^i(t, x) - \int_0^x m^i(x, \xi) \tilde{v}(t, \xi) d\xi \\ &\quad - \int_0^x \int_0^\xi m^i(x, \xi) r^{n+1}(\xi, s) \tilde{v}(t, s) ds d\xi \\ &= \tilde{u}^i(t, x) - \int_0^x m^i(x, \xi) \tilde{v}(t, \xi) d\xi \\ &\quad - \int_0^x \tilde{v}(t, \xi) \int_\xi^x m^i(x, s) r^{n+1}(s, \xi) ds d\xi \\ \tilde{\alpha}^i(t, x) &= \tilde{u}^i(t, x) + \int_0^x r^i(x, \xi) \tilde{v}(t, \xi) d\xi \end{aligned}$$

where we denote, for $i = 1, \dots, n$

$$r^i(x, \xi) = -m^i(x, \xi) - \int_\xi^x m^i(x, s) r^{n+1}(s, \xi) ds$$

D. Main result

The main result regarding the observer design is summarized in the following theorem.

Theorem 3.3: Consider system (10),(11) with boundary conditions (12) and the gains $p_i(\cdot)$, $i = 1, \dots, n$ defined by (27). Then, under the following assumptions

$$\forall i, j = 1, \dots, n \quad \lambda_i, \mu \in C^1([0, 1], \mathbb{R}^*), \quad \sigma_{i,j}, \omega_i, \theta_i \in \mathcal{L}^\infty([0, 1]) \\ \mathbf{u}^0, \mathbf{v}^0 \in \mathcal{L}^2([0, 1])$$

the equilibrium $\tilde{\alpha}^1 \equiv \dots \equiv \tilde{\alpha}^n \equiv \tilde{\beta} \equiv 0$ is exponentially stable in the \mathcal{L}^2 sense.

Proof: By Lemma 3.1, there exists $C, \delta > 0$ such that

$$\begin{aligned} \left\| \left(\tilde{\alpha}^1(t, \cdot), \dots, \tilde{\beta}(t, \cdot) \right) \right\|_{\mathcal{L}^2([0,1])} \\ \leq C \left\| \left(\tilde{\alpha}^1(0, \cdot), \dots, \tilde{\beta}(0, \cdot) \right) \right\|_{\mathcal{L}^2([0,1])} e^{-\delta t} \end{aligned}$$

By Corollary 3.2 and the definition of the inverse transformation kernels, the m^i and r^i , $i = 1, \dots, n+1$ are continuous, thus bounded on the compact set \mathcal{T} . Thus, denoting

$$\|m\|_\infty = \max_{(x,\xi) \in \mathcal{T}, i=1, \dots, n} |m^i(x, \xi)|, \quad \|r\|_\infty = \max_{(x,\xi) \in \mathcal{T}, i=1, \dots, n} |r^i(x, \xi)|$$

we have, from both the direct and inverse transformations (20),(21) and (29),(30) the following inequality

$$\begin{aligned} \left\| \left(\tilde{u}^1(t, \cdot), \dots, \tilde{v}(t, \cdot) \right) \right\|_{\mathcal{L}^2([0,1])} \\ \leq C(1 + \|m\|_\infty)(1 + \|r\|_\infty) \left\| \left(\tilde{u}^1(0, \cdot), \dots, \tilde{v}(0, \cdot) \right) \right\|_{\mathcal{L}^2([0,1])} e^{-\delta t} \end{aligned}$$

which concludes the proof. ■

IV. NUMERICAL SIMULATION

Several industrial processes are modeled by first-order hyperbolic system of the form studied here, such as irrigation channels [3], oil and gas wells [7]. Applying our control design to such processes is the subject of future research requiring a complete physical treatment that is far beyond the space available in this paper. Rather, to illustrate our result, we implement the observer along with the controller designed in [8]. The convergence of the full observer-controller scheme can straightforwardly be proved using a Lyapunov approach very similar to the proof of Theorem 3.3. Further, we implement the resulting output feedback control law on a particularly challenging toy problem corresponding to $n = 2$: we design a system such that the zero equilibrium *cannot* be exponentially stabilized by a *static* output feedback law. Indeed, we choose the in-domain coupling coefficients as follows

$$\begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \omega_1 \\ \sigma_{2,1} & \sigma_{2,2} & \omega_2 \\ \theta_1 & \theta_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

the boundary coefficients as follows

$$q_1 = 1, \quad q_2 = 1.2, \quad \rho_1 = 0, \quad \rho_2 = -0.8$$

and the transport speeds as

$$\lambda_1 = \lambda_2 = \mu = 1$$

This system is a cascade of the (u^2, v) -system into the u^1 system. As proved by [2, Theorem 1], the (u^2, v) -system cannot be stabilized by a static output feedback control law, because the solution η to the following Cauchy problem on $[0, 1]$

$$\eta' = \left| \frac{\omega_2}{\lambda_2} + \frac{\theta_2}{\mu} \eta^2 \right|, \quad \eta(0) = 0, \quad (31)$$

given by

$$\eta(x) = \tan(2x)$$

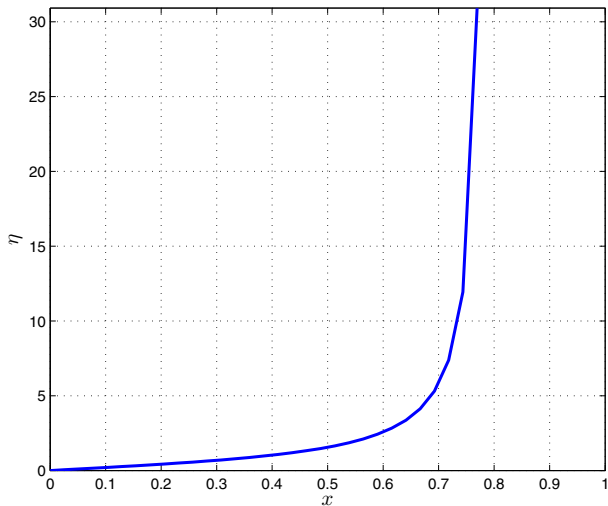


Fig. 3. Maximal solution of the Cauchy problem (31). The solution ceases to exist after $x = \frac{\pi}{4}$.

goes to infinity as x approaches $\frac{\pi}{4}$ as depicted on Figure 3. Equations (1)–(4) are discretized in space and time using an implicit characteristic scheme [6]. The time step is $\Delta t = 0.01$ and the spatial domain $[0, 1]$ is divided in 40 intervals of equal length. Figures 4–6(a) picture the simulation results.

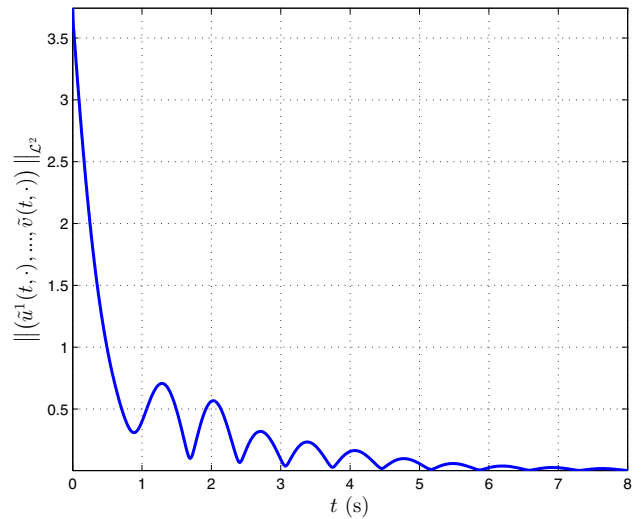
Both the \mathcal{L}^2 norms of the observer error and plant state, plotted on Figure 4, asymptotically tend to zero as time goes to infinity after an initial overshoot.

V. CONCLUSION AND PERSPECTIVES

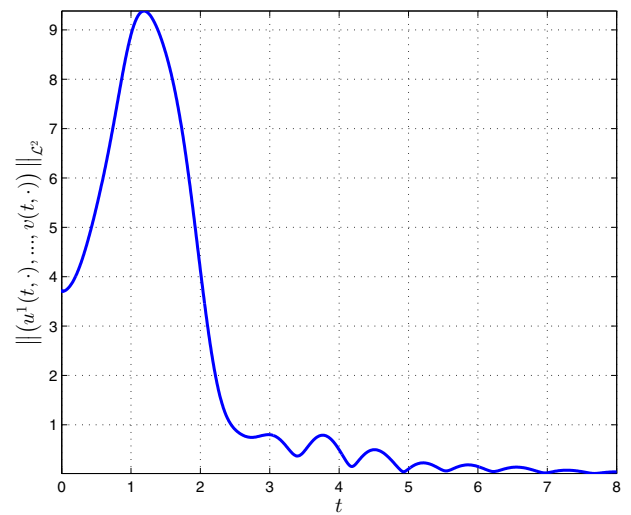
The presented observer can be considered dual of the controller derived in [8], in the sense that the designs lead to the same kernel equations, up to a variable change. Unfortunately, in a lot of applications, the design of a collocated observer is of greater interest, e.g. in the case of multiphase flow control in oil production systems [9]. Such an observer is more difficult to derive using the backstepping method, even when considering measurements of the n states “exiting” at the controlled boundary. The reason for this is the impossibility to add lower triangular integral coupling terms in the target system (as in Equations (14),(16)) when the sensor is located at $x = 1$, because the backstepping transformation is usually *upper* triangular in that case. For the same reason, the generalization of the observer and controller designs to systems with n positive and m negative transport speeds remains an open problem for $m \geq 2$.

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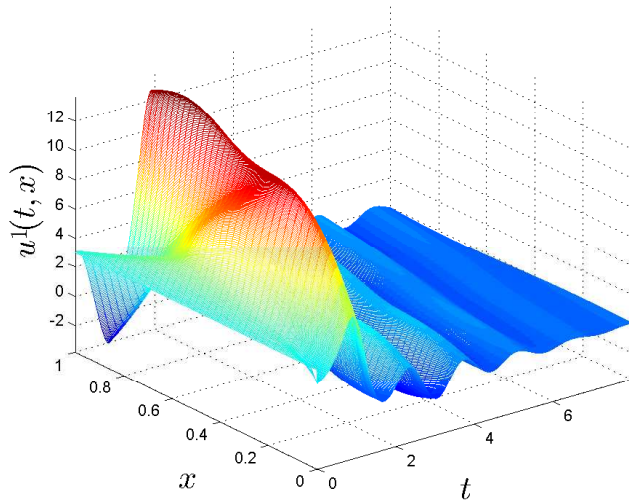
(a)



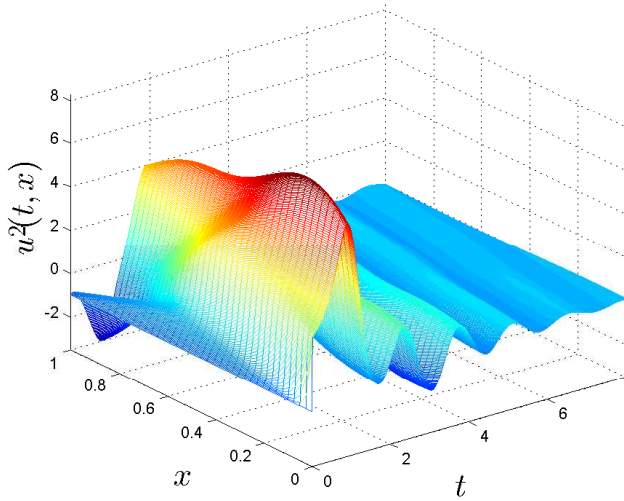
(b)

Fig. 4. Time evolution of the \mathcal{L}^2 norm of the observation error $\|(\tilde{u}^1(t, \cdot), \tilde{u}^2(t, \cdot), \tilde{v}(t, \cdot))\|_{\mathcal{L}^2}$ (Figure 4(a)) and the \mathcal{L}^2 norm of the state $\|(u^1(t, \cdot), u^2(t, \cdot), v(t, \cdot))\|_{\mathcal{L}^2}$ (Figure 4(b)).

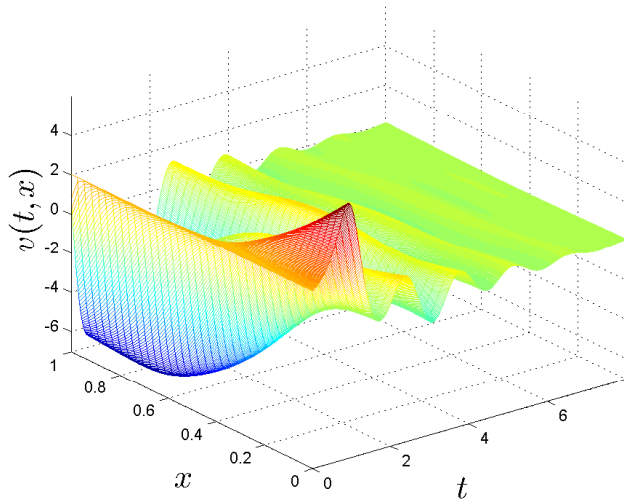
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(a)

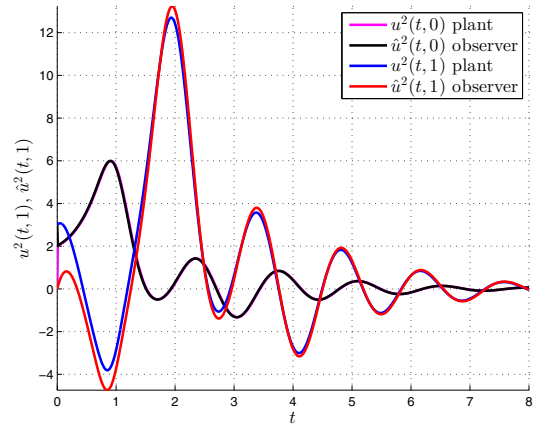


(b)

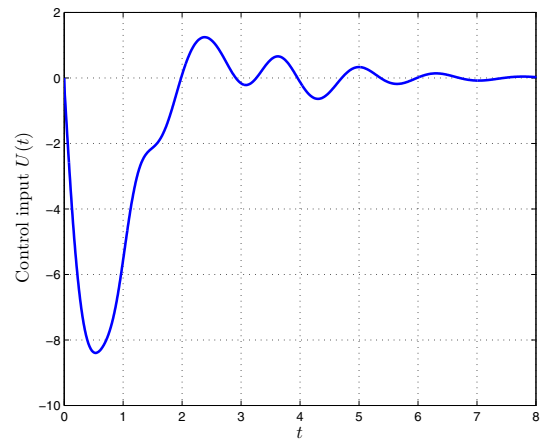


(c)

Fig. 5. Time evolution of the state profiles $u^1(t, x)$ 5(a), $u^2(t, x)$ 5(b) and $v(t, x)$ 5(c).



(a)



(b)

Fig. 6. Comparison of the plant and observer state at both boundaries (Figure 6(a)) and control input (Figure 6(b)).

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