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Brief paper

Minimum time control of heterodirectional linear coupled hyperbolic PDEs[☆]

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ABSTRACT

We solve the problem of stabilizing a general class of linear first-order hyperbolic systems. Considered systems feature an arbitrary number of coupled transport PDEs convecting in either direction. Using the backstepping approach, we derive a full-state feedback law and a boundary observer enabling stabilization by output feedback. Unlike previous results, finite-time convergence to zero is achieved in the theoretical lower bound for control time.

1. Introduction

This article solves the problem of boundary stabilization of a general class of coupled heterodirectional linear first-order hyperbolic systems of Partial Differential Equations (PDEs) in minimum time, with arbitrary numbers m and n of PDEs in each direction and with actuation applied on only one boundary. First-order hyperbolic PDEs are predominant in modeling of traffic flow (Amin, Hante, & Bayen, 2008), heat exchanger (Xu & Sallet, 2002), open channel flow (Coron, d'Andréa Novel, & Bastin, 1999; de Halleux, Prieur, Coron, d'Andréa Novel, & Bastin, 2003) or multiphase flow (Di Meglio, 2011; Djordjevic, Bosgra, Van den Hof, & Jeltsema, 2010; Dudret, Beauchard, Ammouri, & Rouchon, 2012). Research on controllability and stability of hyperbolic systems has first focused on explicit computation of the solution along the characteristic curves in the framework of the C^1 norm (Greenberg & Tsien, 1984; Li, 1994; Qin, 1985). Later, Control Lyapunov Functions methods emerged, enabling the design of dissipative boundary conditions for nonlinear hyperbolic systems (Coron, 2009; Coron, Bastin, & d'Andréa Novel, 2008). In Coron, Vazquez, Krstic, and Bastin (2013) control laws for a system of two coupled nonlinear PDEs are derived, whereas in Castillo, Witrant, Prieur, and Dugard (2012), Coron et al. (2008), Prieur and Mazenc (2012), Prieur, Winkin, and Bastin (2008), Santos and Prieur (2008) sufficient conditions for exponential stability are given for various classes of quasilinear first-order hyperbolic system. These conditions typically impose restrictions on the magnitude of the coupling coefficients.

In Coron et al. (2013) a backstepping transformation is used to design a single boundary output-feedback controller. This control law yields H^2 exponential stability of closed loop 2-state heterodirectional linear and quasilinear hyperbolic system for arbitrary large coupling coefficients. A similar approach is used in Di Meglio, Vazquez, and Krstic (2013) to design output feedback laws for a system of coupled first-order hyperbolic linear PDEs with $m = 1$ controlled negative velocity and n positive ones. The generalization of this result to an arbitrary number m of controlled negative velocities is presented in Hu, Di Meglio, Vazquez, and Krstic (2015). There, the proposed control law yields finite-time convergence to zero, but the convergence time is larger than the minimum control time, derived in Li and Rao (2010) and Woittennek, Rudolph, and Knüppel (2009). This is due to the presence of non-local coupling terms in the targeted closed-loop behavior. The main contribution of this paper is a minimum time stabilizing controller. More precisely, a proposed boundary feedback law ensures finite-time convergence of all states to zero in minimum-time. This minimum-time, defined in Li and Rao (2010), Woittennek et al. (2009) is the sum of the two largest time of transport in each direction.

Our approach is the following. Using a backstepping approach (with a Volterra transformation) the system is mapped to a *target* system with desirable stability properties. This target system is a copy of the original dynamics with a modified in-domain coupling structure. More precisely, the target system is

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designed as an exponentially stable cascade. A full-state feedback law guaranteeing exponential stability of the zero equilibrium in the $\mathcal{L}^2$-norm is then designed. This full-state feedback law requires full distributed measurements. For this reason we derive a boundary observer relying on measurements of the states at a single boundary (the anti-collocated one). Similarly to the control design, the observer error dynamics are mapped to a target system using a Volterra transformation. Along with the full-state feedback law, this yields an output feedback controller amenable to implementation.

Technically, this design poses a novel challenge as far as proving the well-posedness of the Volterra transformation. The transformation kernels satisfy a system of equations with a cascade structure akin to the target system one. This structure enables a recursive proof of existence of the transformation kernels using tools similar to the ones presented in Hu et al. (2015).

The paper is organized as follows. In Section 2 we introduce the model equations and the notations. In Section 3 we present the stabilization result: the target system and its properties are presented in Section 3.1. In Section 3.2 we derive the backstepping transformation. Section 4 contains the main technical difficulty of this paper which is the proof of well-posedness of the kernel equations. In Section 4.1 we transform the kernel equations into an integral equation using the method of characteristics. In Section 4.2 we solve the integral equations using the method of successive approximations. In Section 5 we present the control feedback law and its properties. In Section 6 we present the uncollocated observer design. In Section 7 we give some simulation results. Finally in Section 8 we give some concluding remarks

2. Problem description

2.1. System under consideration

We consider the following general linear hyperbolic system which appears in Saint-Venant equations, heat exchangers equations and other linear hyperbolic balance laws (see Bastin & Coron, 2015).

$$u_t(t,x) + \Lambda^+ u_x(t,x) = \Sigma^{++}u(t,x) + \Sigma^{+-}v(t,x) \tag{1}$$

$$v_t(t,x) - \Lambda^- v_x(t,x) = \Sigma^{-+}u(t,x) + \Sigma^{--}v(t,x) \tag{2}$$

evolving in $\{(t,x)|\ t > 0,\ x \in [0,1]\}$, with the following linear boundary conditions

$$u(t,0) = Q_0 v(t,0), \qquad v(t,1) = R_1 u(t,1) + U(t) \tag{3}$$

where

$$u = (u_1 \dots u_n)^T, \qquad v = (v_1 \dots v_m)^T \tag{4}$$

$$\Lambda^+ = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}, \qquad \Lambda^- = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix} \tag{5}$$

with constant speeds:

$$-\mu_m < \dots < -\mu_1 < 0 < \lambda_1 \le \dots \le \lambda_n \tag{6}$$

and constant real coupling matrices as well as the feedback control input

$$\Sigma^{++} = \{\sigma_{ij}^{++}\}_{1\le i\le n, 1\le j\le n} \qquad \Sigma^{+-} = \{\sigma_{ij}^{+-}\}_{1\le i\le n, 1\le j\le m} \tag{7}$$

$$\Sigma^{-+} = \{\sigma_{ij}^{-+}\}_{1\le i\le m, 1\le j\le n} \qquad \Sigma^{--} = \{\sigma_{ij}^{--}\}_{1\le i\le m, 1\le j\le m} \tag{8}$$

$$Q_0 = \{q_{ij}\}_{1\le i\le n, 1\le j\le m} \qquad R_1 = \{\rho_{ij}\}_{1\le i\le m, 1\le j\le n}. \tag{9}$$

The initial conditions denoted u_0 and v_0 are assumed to belong to $L^2([0,1],\Re)$.

Remark 1. The coupling terms are assumed constant here but the results of this paper can be adjusted for spatially-varying coupling terms.

2.2. Control problem

The goal is to design feedback control inputs $U(t) = (U_1(t), \dots, U_m(t))^T$ such that the zero equilibrium is reached in minimum time $t = t_F$, where

$$t_F = \frac{1}{\mu_1} + \frac{1}{\lambda_1}. \tag{10}$$

This problem is very similar to the one presented in Hu et al. (2015). The main difference is that the time proposed in this paper in which the controlled system is stabilized is much smaller.

3. Control design

The control design is based on the backstepping approach: using a Volterra transformation, we map the system (1)–(3) to a target system with desirable properties of stability.

3.1. Target system design

We map the system (1)–(3) to the following system

$$\alpha_t(t,x) + \Lambda^+\alpha_x(t,x) = \Sigma^{++}\alpha(t,x) + \Sigma^{+-}\beta(t,x) + \int_0^x C^+(x,\xi)\alpha(t,\xi)d\xi + \int_0^x C^-(x,\xi)\beta(t,\xi)d\xi \tag{11}$$

$$\beta_t(t,x) - \Lambda^-\beta_x(t,x) = \Omega(x)\beta(t,x) \tag{12}$$

with the following boundary conditions

$$\alpha(t,0) = Q_0\beta(t,0) \qquad \beta(t,1) = 0 \tag{13}$$

where C^+ and C^- are L^∞ matrix functions on the domain

$$\mathcal{T} = \{0 \le \xi \le x \le 1\} \tag{14}$$

while $\Omega \in L^\infty(0,1)$ is an upper triangular matrix with the following structure

$$\Omega(x) = \begin{pmatrix} \omega_{1,1}(x) & \omega_{1,2}(x) & \cdots & \omega_{1,m}(x) \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \omega_{m-1,m-1}(x) & \omega_{m-1,m}(x) \\ 0 & \cdots & 0 & \omega_{m,m}(x) \end{pmatrix}. \tag{15}$$

This system is designed as a copy of the original dynamics, from which the coupling terms of (2) are removed. The integral coupling appearing in (11) is added for the control design but does not have any incidence on the stability of the target system: since all the velocities are strictly positive the integral terms are feedforward terms.

Remark 2. This new target system is the main difference with Hu et al. (2015) and is the innovative aspect of this paper.

Remark 3. Without any loss of generality, one can assume that $\forall 1 \le i \le m, \sigma_{ii}^{--} = 0$ (such coupling terms can be removed using a change of coordinates as presented in, e.g., Coron et al., 2013). In this case, $\Omega(x)$ has exactly the same structure as the matrix $G(x)$ in Hu et al. (2015).

Besides, the following lemma assesses the finite-time stability of the target system.

Lemma 1. *The system (11), (12) reaches its zero equilibrium in finite-time $t_F = \frac{1}{\mu_1} + \frac{1}{\lambda_1}$*

Proof. The proof of this lemma is straightforward using the proof of Hu et al. (2015, Lemma 3.1). The system is a cascade of $\tilde{\alpha}$-system (that has zero input at the left boundary) into the β-system (that has zero input at the right boundary once $\tilde{\alpha}$ becomes null).

Remark 4. The zero equilibrium of (11)–(12) with boundary conditions (13) and initial conditions $(\alpha^0, \beta^0) \in \mathcal{L}^2([0, 1])$ is exponentially stable in the $\mathcal{L}^2$ sense. This can be proved using the fact that for initial condition in L^2, the solution stays in L^2, and becomes identically null in finite time.

3.2. Volterra transformation

In order to map the original system (1)–(3) to the target system (11)–(13), we use the following Volterra transformation

$$\alpha(t,x) = u(t,x) \tag{16}$$

$$\beta(t,x) = v(t,x) - \int_0^x (K(x,\xi)u(\xi) + L(x,\xi)v(\xi))d\xi \tag{17}$$

where the kernels K and L, defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2 | \xi \leq x\}$ have yet to be defined. Differentiating (17) with respect to space and using the Leibniz rule yields

$$\begin{aligned}\beta_x(t,x) &= v_x(t,x) - K(x,x)u(t,x) - L(x,x)v(t,x) \\ &\quad - \int_0^x K_x(x,\xi)u(t,\xi) + L_x(x,\xi)v(t,\xi)d\xi.\end{aligned} \tag{18}$$

Differentiating with respect to time, using (1), (2) and integrating by parts yields

$$\begin{aligned}\beta_t(t,x) &= \Lambda^- v_x(t,x) + \Sigma^{-+}u(t,x) + \Sigma^{--}v(t,x) \\ &- \int_0^x \Big[K(x,\xi)\Sigma^{++}u(t,\xi) + K(x,\xi)\Sigma^{+-}v(t,\xi) \\ &+ L(x,\xi)\Sigma^{-+}u(t,\xi) + L(x,\xi)\Sigma^{--}v(t,\xi)\Big]d\xi \\ &+ K(x,x)\Lambda^+u(t,x) - K(x,0)\Lambda^+u(t,0) \\ &- L(x,x)\Lambda^-v(t,x) + L(x,0)\Lambda^-v(t,0) \\ &- \int_0^x \big[K_\xi(x,\xi)\Lambda^+u(t,\xi) - L_\xi(x,\xi)\Lambda^-v(t,\xi)\big]d\xi.\end{aligned} \tag{19}$$

Plugging these expressions into the target system (11)–(13), taking $x = 0$ in (17) and using the corresponding boundary conditions (3) yields the following system of kernel equations

$$0 = \Sigma^{-+} + K(x,x)\Lambda^+ + \Lambda^- K(x,x) \tag{20}$$

$$0 = \Sigma^{--} + \Lambda^- L(x,x) - L(x,x)\Lambda^- - \Omega(x) \tag{21}$$

$$0 = K(x,0)\Lambda^+ Q_0 - L(x,0)\Lambda^- \tag{22}$$

$$\begin{aligned}0 &= \Lambda^- K_x(x,\xi) - K_\xi(x,\xi)\Lambda^+ - K(x,\xi)\Sigma^{++} \\ &\quad - L(x,\xi)\Sigma^{-+} + \Omega(x)K(x,\xi)\end{aligned} \tag{23}$$

$$\begin{aligned}0 &= \Lambda^- L_x(x,\xi) + L_\xi(x,\xi)\Lambda^- - L(x,\xi)\Sigma^{--} \\ &\quad - K(x,\xi)\Sigma^{+-} + \Omega(x)L(x,\xi).\end{aligned} \tag{24}$$

We get the following equations for $C^-(x,\xi)$ and $C^+(x,\xi)$

$$C^-(x,\xi) = \Sigma^{+-}L(x,\xi) + \int_\xi^x C^-(x,s)L(s,\xi)ds \tag{25}$$

$$C^+(x,\xi) = \Sigma^{+-}K(x,\xi) + \int_\xi^x C^-(x,s)K(s,\xi)ds. \tag{26}$$

Remark 5. Similarly to Hu et al. (2015, Remark 3), one can notice that for each $x \in [0, 1]$, Eq. (25) is a Volterra equation on $[0, x]$ where $C^-(x,\cdot)$ is the unknown. Assuming that K and L are well defined and bounded, so is C^-. Using (26) yields explicitly C^+ as a function of C^- and K.

Developing Eqs. (20)–(24) we get the following set of kernel PDEs:

for $1 \leq i \leq m$, $1 \leq j \leq n$

$$\begin{aligned}\mu_i\partial_x K_{ij}(x,\xi) - \lambda_j\partial_\xi K_{ij}(x,\xi) &= \sum_{k=1}^n \sigma_{kj}^{++}K_{ik}(x,\xi) \\ + \sum_{p=1}^m \sigma_{pj}^{-+}L_{ip}(x,\xi) &- \sum_{i\leq p\leq m} K_{pj}(x,\xi)\omega_{ip}(x)\end{aligned} \tag{27}$$

for $1 \leq i \leq m$, $1 \leq j \leq m$

$$\begin{aligned}\mu_i\partial_x L_{ij}(x,\xi) + \mu_j\partial_\xi L_{ij}(x,\xi) &= \sum_{k=1}^m \sigma_{kj}^{--}L_{ik}(x,\xi) \\ + \sum_{p=1}^n \sigma_{pj}^{+-}K_{ip}(x,\xi) &- \sum_{i\leq p\leq m} L_{pj}(x,\xi)\omega_{ip}(x)\end{aligned} \tag{28}$$

with the following set of boundary conditions

$$\forall 1 \leq i \leq m,\ \forall j \leq n, \quad K_{ij}(x,x) = -\frac{\sigma_{ij}^{-+}}{\mu_i + \lambda_j} = k_{ij} \tag{29}$$

$$\forall 1 \leq i,j \leq m,\ j < i \quad L_{ij}(x,x) = \frac{-\sigma_{ij}^{--}}{\mu_i - \mu_j} \tag{30}$$

$$\forall 1 \leq i,j \leq m, \quad \mu_j L_{ij}(x,0) = \sum_{k=1}^n \lambda_k K_{ik}(x,0)q_{kj}. \tag{31}$$

Besides, (21) imposes

$$\forall i \leq j \quad \omega_{ij}(x) = (\mu_i - \mu_j)L_{ij}(x,x) + \sigma_{ij}^{--}. \tag{32}$$

This induces a coupling between the kernels through Eqs. (27) and (28) that could appear as nonlinear at first sight. However, as it will appear in the proof of the following theorem, the coupling has a linear cascade structure. More precisely, the well-posedness of the target system is assessed in the following theorem.

Theorem 1. *Consider system (27)–(31). There exists a unique solution K and L in $L^\infty(\mathcal{T})$.*

The proof of this theorem is described in the following section and uses the cascade structure of the kernel equations (which is due to the particular shape of the matrix Ω).

4. Well-posedness of the kernel equation

To prove the well-posedness of the kernel equations we classically (see John, 1960 and Whitham, 2011) transform the kernel equations into integral equations and use the method of successive approximations.

By induction, let us consider the following property $P(s)$ defined for all $1 \leq s \leq m$:

$\forall 1 \leq j \leq n, \forall 1 \leq l \leq m$ and $\forall m + 1 - s \leq i \leq m$ the problem (27)–(31) where Ω is defined by (32) has a unique solution $K_{ij}(\cdot,\cdot)$, $L_{il}(\cdot,\cdot) \in L^\infty(\mathcal{T})$.

Initialization: For $s = 1$, system (27)–(31) rewrites as follows

for $1 \leq j \leq n$

$$\mu_m\partial_x K_{mj} - \lambda_j\partial_\xi K_{mj} = \sum_{k=1}^n \sigma_{kj}^{++}K_{mk}(x,\xi) \tag{33}$$

$$+ \sum_{p=1}^m \sigma_{pj}^{-+}L_{mp}(x,\xi) - K_{mj}(x,\xi)\sigma_{mm}^{--} \tag{34}$$

for $1 \le j \le m$

$$\mu_m \partial_x L_{mj} + \mu_j \partial_\xi L_{mj} = \sum_{k=1}^{m} \sigma_{kj}^{--} L_{mk}(x,\xi) \tag{35}$$

$$+ \sum_{p=1}^{n} \sigma_{pj}^{+-} K_{mp}(x,\xi) - L_{mj}(x,\xi)\sigma_{mm}^{--} \tag{36}$$

with the following set of boundary conditions

$$\forall 1 \le j \le n, \quad K_{mj}(x,x) = -\frac{\sigma_{mj}^{-+}}{\mu_m + \lambda_j} = k_{mj} \tag{37}$$

$$\forall 1 \le j < m, \quad L_{mj}(x,x) = -\frac{\sigma_{mj}^{--}}{\mu_m - \mu_j} \tag{38}$$

$$\forall 1 \le j \le m, \quad \mu_j L_{mj}(x,0) = \sum_{k=1}^{n} \lambda_k K_{mk}(x,0) q_{kj}. \tag{39}$$

The well-posedness of such system is quite straightforward using (Hu et al., 2015).

Induction: Let us assume that the property $P(s-1)$ $(1 < s \le m-1)$ is true. We consequently have that $\forall m+2-s \le p \le m, \forall 1 \le j \le n$, $\forall 1 \le l \le m$ $K_{pj}(\cdot,\cdot)$ and $L_{pl}(\cdot,\cdot)$ are bounded. In the following we take $i = m+1-s$. We now show that (27)–(31) is well-posed and that $K_{ij}(\cdot,\cdot)$ and $L_{il}(\cdot,\cdot) \in L^\infty(\mathcal{T})$.

4.1. Method of characteristics

4.1.1. Characteristics of the K kernels

For each $1 \le j \le n$ and $(x,\xi) \in \mathcal{T}$, we define the following characteristic lines $(x_{ij}(x,\xi,\cdot), \xi_{ij}(x,\xi,\cdot))$ corresponding to Eq. (27)

$$\begin{cases} \dfrac{dx_{ij}}{ds}(x,\xi,s) = -\mu_i & s \in [0, s_{ij}^F(x,\xi)] \\ x_{ij}(x,\xi,0) = x, & x_{ij}(x,\xi,s_{ij}^F(x,\xi)) = x_{ij}^F(x,\xi) \end{cases} \tag{40}$$

$$\begin{cases} \dfrac{d\xi_{ij}}{ds}(x,\xi,s) = \lambda_j & s \in [0, s_{ij}^F(x,\xi)] \\ \xi_{ij}(x,\xi,0) = \xi, & \xi_{ij}(x,\xi,s_{ij}^F(x,\xi)) = x_{ij}^F(x,\xi). \end{cases} \tag{41}$$

These lines originate at the point (x,ξ) and terminate on the hypotenuse at the point $(x_{ij}^F(x,\xi), x_{ij}^F(x,\xi))$. Integrating (27) along these characteristics and using the boundary conditions (29) we get

$$\begin{aligned} K_{ij}(x,\xi) &= k_{ij} \\ &+ \int_0^{s_{ij}^F(x,\xi)} \Big[\sum_{k=1}^{n} \sigma_{kj}^{++} K_{ik}(x_{ij}(x,\xi,s), \xi_{ij}(x,\xi,s)) \\ &+ \sum_{k=1}^{m} \sigma_{kj}^{-+} L_{ik}(x_{ij}(x,\xi,s), \xi_{ij}(x,\xi,s)) \\ &- \sum_{i \le p \le m} K_{pj}(x_{ij}(x,\xi,s), \xi_{ij}(x,\xi,s)) \cdot ((\mu_i - \mu_p) \\ &\times L_{ip}(x_{ij}(x,\xi,s), x_{ij}(x,\xi,s)) + \sigma_{ip}^{--}) \Big] ds. \end{aligned} \tag{42}$$

We can notice that the last sum uses the expression of K_{pj} for $i \le p \le m$. This term is known and bounded for $p > i$ (induction assumption). For $p = i$, $\mu_i = \mu_p$ and the term $(\mu_i - \mu_p) L_{ip}(x_{ij}(x,\xi,s), x_{ij}(x,\xi,s))$ cancels.

4.1.2. Characteristics of the L kernels

For each $1 \le j \le n$ and $(x,\xi) \in \mathcal{T}$, we define the following characteristic lines $(\chi_{ij}(x,\xi,\cdot), \zeta_{ij}(x,\xi,\cdot))$ corresponding to Eq. (28)

$$\begin{cases} \dfrac{d\chi_{ij}}{d\nu}(x,\xi,s) = -\mu_i & \nu \in [0, \nu_{ij}^F(x,\xi)] \\ \chi_{ij}(x,\xi,0) = x, & \chi_{ij}(x,\xi,\nu_{ij}^F(x,\xi)) = \chi_{ij}^F(x,\xi) \end{cases} \tag{43}$$

$$\begin{cases} \dfrac{d\zeta_{ij}}{d\nu}(x,\xi,s) = -\mu_j & \nu \in [0, \nu_{ij}^F(x,\xi)] \\ \zeta_{ij}(x,\xi,0) = \xi, & \zeta_{ij}(x,\xi,\nu_{ij}^F(x,\xi)) = \zeta_{ij}^F(x,\xi). \end{cases} \tag{44}$$

These lines all originate from (x,ξ) and terminate at the point $(\chi_{ij}^F(x,\xi), \zeta_{ij}^F(x,\xi))$, i.e. either at $(\chi_{ij}^F(x,\xi), \chi_{ij}^F(x,\xi))$ or at $(\chi_{ij}^F(x,\xi), 0)$. Integrating (28) along these characteristic and using the boundary conditions (30), (31) yields

$$\begin{aligned} L_{ij}(x,\xi) &= -\delta_{ij}(x,\xi)\frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} \\ &+ (1-\delta_{ij})\frac{1}{\mu_j}\sum_{k=1}^{n} \lambda_k q_{kj} K_{ik}(\chi_{ij}^F(x,\xi), 0) \\ &+ \int_0^{\nu_{ij}^F(x,\xi)} \Big[\sum_{p=1}^{m} \sigma_{pj}^{--} L_{ip}(\chi_{ij}(x,\xi,\nu), \zeta_{ij}(x,\xi,\nu)) \\ &+ \sum_{k=1}^{n} \sigma_{kj}^{+-} K_{ik}(\chi_{ij}(x,\xi,\nu), \zeta_{ij}(x,\xi,\nu)) \\ &- \sum_{i \le p \le m} L_{pj}(\chi_{ij}(x,\xi,\nu), \zeta_{ij}(x,\xi,\nu))((\mu_i - \mu_p) \\ &\times L_{ip}(\chi_{ij}(x,\xi,\nu), \chi_{ij}(x,\xi,\nu)) + \sigma_{ip}^{--}) \Big] d\nu \end{aligned} \tag{45}$$

where the coefficient $\delta_{ij}(x,\xi)$ is defined by

$$\delta_{i,j}(x,\xi) = \begin{cases} 1 & \text{if } j < i \quad \text{and} \quad \mu_i \xi - \mu_j x \ge 0 \\ 0 & \text{else.} \end{cases} \tag{46}$$

This coefficient reflects the facts that, as mentioned above, some characteristics terminate on the hypotenuse and others on the axis $\xi = 0$. We can now plug (42) evaluated at $(\chi_{ij}^F(x,\xi), 0)$ into (45) which yields

$$\begin{aligned} L_{ij}(x,\xi) &= -\delta_{ij}(x,\xi)\frac{\sigma_{ij}^{--}}{\mu_i - \mu_j} + (1-\delta_{ij})\frac{1}{\mu_j}\sum_{k=1}^{n} \lambda_k q_{kj} k_{ik} \\ &+ (1-\delta_{ij})\frac{1}{\mu_j}\sum_{r=1}^{n} \lambda_r q_{rj} \int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)} \\ &\times \Big[\sum_{k=1}^{n} \sigma_{kr}^{++} K_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),0,s), \xi_{ir}(\chi_{ij}^F(x,\xi),0,s)) \\ &+ \sum_{k=1}^{m} \sigma_{kr}^{-+} L_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),0,s), \xi_{ir}(\chi_{ij}^F(x,\xi),0,s)) \\ &- \sum_{i \le p \le m} K_{pr}(x_{ir}(\chi_{ij}^F(x,\xi),0,s), \xi_{ir}(\chi_{ij}^F(x,\xi),0,s)) \\ &\cdot ((\mu_i - \mu_p) L_{ip}(x_{ir}(\chi_{ij}^F(x,\xi),0,s), x_{ir}(\chi_{ij}^F(x,\xi),0,s)) \\ &+ \sigma_{ip}^{--}) \Big] ds + \int_0^{\nu_{ij}^F(x,\xi)} \Big[\sum_{p=1}^{m} \sigma_{pj}^{--} L_{ip}(\chi_{ij}(x,\xi,\nu), \\ &\zeta_{ij}(x,\xi,\nu)) + \sum_{k=1}^{n} \sigma_{kj}^{+-} K_{ik}(\chi_{ij}(x,\xi,\nu), \zeta_{ij}(x,\xi,\nu)) \\ &- \sum_{i \le p \le m} L_{pj}(\chi_{ij}(x,\xi,\nu), \zeta_{ij}(x,\xi,\nu)) \\ &\cdot ((\mu_i - \mu_p) L_{ip}(\chi_{ij}(x,\xi,\nu), \chi_{ij}(x,\xi,\nu)) + \sigma_{ip}^{--}) \Big] d\nu. \end{aligned} \tag{47}$$

4.2. Method of successive approximations

In order to solve the integral equations (42), (47) we use the method of successive approximations. We define

$$\forall 1\le j\le n\quad \phi_j^1(x,\xi)=k_{ij}-\int_0^{s_{ij}^F(x,\xi)}\sum_{i<p\le m}K_{pj}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\sigma_{ip}^{--}ds \tag{48}$$

$$\begin{aligned}\forall 1\le j\le m\quad \phi_j^2(x,\xi)&=-\delta_{ij}(x,\xi)\frac{\sigma_{ij}^{--}}{\mu_i-\mu_j}+(1-\delta_{ij})\frac{1}{\mu_j}\\&\times\sum_{k=1}^n\lambda_k q_{kj}k_{ik}-(1-\delta_{ij})\frac{1}{\mu_j}\sum_{r=1}^n\lambda_r q_{rj}\int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)}\\&\sum_{i<p\le m}K_{pr}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\sigma_{ip}^{--}\\&-\int_0^{\nu_{ij}^F(x,\xi)}\sum_{i<p\le m}L_{pj}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\sigma_{ip}^{--}d\nu.\end{aligned} \tag{49}$$

Besides we denote H as the vector containing the kernels

$$H=\begin{pmatrix}K_{i1} & \cdots & K_{in} & L_{i1} & \cdots & L_{im}\end{pmatrix}^\top \tag{50}$$

$$\Psi=\begin{pmatrix}\phi_1^1 & \cdots & \phi_n^1 & \phi_1^2 & \cdots & \phi_m^2\end{pmatrix}^\top. \tag{51}$$

We now consider the following operators: $\forall 1\le j\le n$

$$\begin{aligned}\Phi_j^1(H)(x,\xi)=\int_0^{s_{ij}^F(x,\xi)}\Bigg[&\sum_{k=1}^n\sigma_{kj}^{++}K_{ik}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\\&+\sum_{k=1}^m\sigma_{kj}^{-+}L_{ik}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\\&-\sum_{i<p\le m}K_{pj}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\\&\cdot((\mu_i-\mu_p)L_{ip}(x_{ij}(x,\xi,s),x_{ij}(x,\xi,s)))\\&+\sigma_{ii}^{--}K_{ij}(x_{ij}(x,\xi,s),\xi_{ij}(x,\xi,s))\Bigg]ds\end{aligned} \tag{52}$$

$\forall 1\le j\le n$

$$\begin{aligned}\Phi_j^2(H)(x,\xi)=(1-\delta_{ij})\frac{1}{\mu_j}\sum_{r=1}^n\lambda_r q_{rj}&\int_0^{s_{ir}^F(\chi_{ij}^F(x,\xi),0)}\\\times\Bigg[&\sum_{k=1}^n\sigma_{kr}^{++}K_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\\&+\sum_{k=1}^m\sigma_{kr}^{-+}L_{ik}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\\&-\sum_{i<p\le m}K_{pr}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\\&\cdot((\mu_i-\mu_p)L_{ip}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),x_{ir}(\chi_{ij}^F(x,\xi),0,s)))\\&-K_{ir}(x_{ir}(\chi_{ij}^F(x,\xi),0,s),\xi_{ir}(\chi_{ij}^F(x,\xi),0,s))\sigma_{ii}^{--}\Bigg]ds\\+\int_0^{\nu_{ij}^F(x,\xi)}\Bigg[&\sum_{p=1}^m\sigma_{pj}^{--}L_{ip}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\\&+\sum_{k=1}^n\sigma_{kj}^{+-}K_{ik}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\\&-\sum_{i<p\le m}L_{pj}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\\&\cdot((\mu_i-\mu_p)L_{ip}(\chi_{ij}(x,\xi,\nu),\chi_{ij}(x,\xi,\nu)))\\&-L_{ij}(\chi_{ij}(x,\xi,\nu),\zeta_{ij}(x,\xi,\nu))\sigma_{ii}^{--}\Bigg]d\nu.\end{aligned} \tag{53}$$

We set $\Phi[H](x,\xi)=[\Phi^1[H](x,\xi)^T,\Phi^2[H](x,\xi)^\top]^\top$ We define the following sequence

$$H^0(x,\xi)=0 \tag{54}$$

$$H^q(x,\xi)=\Psi(x,\xi)+\Phi(H^{q-1})(x,\xi). \tag{55}$$

Consequently, if the sequence H^q has a limit, then this limit is a solution of the integral equation and therefore of the original system.

We define the increment $\Delta H^q=H^q-H^{q-1}$ (with $\Delta H^0=\Psi$). Provided the limit exists one has

$$H(x,\xi)=\lim_{q\to+\infty}H^q(x,\xi)=\sum_{q=0}^{+\infty}\Delta H^q(x,\xi). \tag{56}$$

We now prove the convergence of the series.

4.3. Convergence of the successive approximation series

Similarly to Di Meglio et al. (2013), Hu et al. (2015) we want to find a recursive upper bound in order to prove the convergence of the series. We first define

$$\bar\Phi=\max_j\max_{(x,\xi)\in\mathcal{T}}\{|\phi_{i,j}^1(x,\xi)|,|\phi_{ij}^2(x,\xi)|\} \tag{57}$$

$$\bar\sigma=\max_{k,j}\{\sigma_{kj}^{++},\sigma_{kj}^{+-},\sigma_{kj}^{-+},\sigma_{kj}^{--}\},\qquad \bar q=\max_{k,j}\{q_{kj}\}$$

$$\bar\mu=\max_p\{|\mu_i-\mu_p|\},\qquad \bar\lambda=\max\{\lambda_n,\mu_n\}$$

$$\tilde\lambda=\max\left\{\frac{1}{\lambda_1},\frac{1}{\mu_1}\right\},\qquad M_\lambda=\max_{j=1,\dots,m}\left\{\frac{1}{\mu_j}\right\}.$$

We then define $\bar S=\max_{p>i,1\le j\le n}\{\|K_{pj}\|,\|L_{pj}\|\}$ which is well defined according to the hypothesis $P(s-1)$. Moreover we set

$$M=(n\bar\lambda\tilde\lambda\bar q+1)[(n+m+1)\bar\sigma+m\bar\mu\bar S]M_\lambda. \tag{58}$$

Lemma 2. *Assume that for some $1\le q$, one has, for all $(x,\xi)\in\mathcal{T}$*

$$\forall j=1,\dots,m+n\quad |\Delta H_j^q(x,\xi)|\le\bar\Phi\frac{M^qx^q}{q!} \tag{59}$$

where $\Delta H_j^q(x,\xi)$ is the jth component of $\Delta H^q(x,\xi)$.

Then, one has

$$\forall j=1,\dots,m+n\quad |\Delta H_j^{q+1}(x,\xi)|\le\bar\Phi\frac{M^{q+1}x^{q+1}}{(q+1)!}. \tag{60}$$

The proof of this lemma is quite straightforward using the ones presented in Di Meglio et al. (2013), Vazquez, Coron, Krstic, and Bastin (2011) since all the characteristic lines have the same direction along the x-axis. Consequently, using similar methods as the ones presented in Di Meglio et al. (2013), Vazquez et al. (2011), we get that (56) converges and thus the property $P(s)$ is true. This concludes the proof by induction of Theorem 1.

5. Control law and main results

We now state the main stabilization result as follows.

Theorem 2. *System (1)–(2) with boundary conditions (3) and the following feedback control law*

$$U(t)=-R_1u(t,1)+\int_0^1[K(1,\xi)u(t,\xi)+L(1,\xi)v(t,\xi)]d\xi \tag{61}$$

reaches its zero equilibrium in finite time t_F where t_F is given by (10). The zero equilibrium is exponentially stable in the L^2-sense.

Proof. Notice first that evaluating (17) at $x = 1$ yields (61). Besides, rewrite (17) as follows

$$\begin{pmatrix}\alpha(t,x)\\ \beta(t,x)\end{pmatrix} = \begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} - \int_0^x \begin{pmatrix}0 & 0\\ K(x,\xi) & L(x,\xi)\end{pmatrix}\begin{pmatrix}u(t,\xi)\\ v(t,\xi)\end{pmatrix} d\xi. \tag{62}$$

It is a classical Volterra equation of the second kind. One can check from Hochstadt (2011) that there exists a unique function $\mathcal{S}$ such that

$$\begin{pmatrix}u(t,x)\\ v(t,x)\end{pmatrix} = \begin{pmatrix}\alpha(t,x)\\ \beta(t,x)\end{pmatrix} - \int_0^x \mathcal{S}(x,\xi)\begin{pmatrix}\alpha(t,\xi)\\ \beta(t,\xi)\end{pmatrix} d\xi. \tag{63}$$

Applying Lemma 2 implies that (α,β) go to zero in finite time t_F, therefore (u,v) converge to zero in finite time.

6. Uncollocated observer design and output feedback controller

In this section we design an observer that relies on the measurements of v at the left boundary, i.e. we measure

$$y(t) = v(t,0). \tag{64}$$

Then, using the estimates given by our observer and the control law (61), we derive an output feedback controller.

6.1. Observer design

The observer equations read as follows

$$\begin{aligned}\hat{u}_t(t,x) + \Lambda^+\hat{u}_x(t,x) &= \Sigma^{++}\hat{u}(t,x) + \Sigma^{+-}\hat{v}(t,x)\\ &\quad - P^+(x)(\hat{v}(t,0) - v(t,0)) \qquad (65)\\ \hat{v}_t(t,x) + \Lambda^-\hat{v}_x(t,x) &= \Sigma^{-+}\hat{u}(t,x) + \Sigma^{--}\hat{v}(t,x)\\ &\quad - P^-(x)(\hat{v}(t,0) - v(t,0)) \qquad (66)\end{aligned}$$

with the boundary conditions

$$\hat{u}(t,0) = Q_0 v(t,0), \qquad \hat{v}(t,1) = R_1\hat{u}(t,1) + U(t) \tag{67}$$

where $P^+(\cdot)$ and $P^-(\cdot)$ have yet to be designed. This yields the following error system

$$\begin{aligned}\tilde{u}_t(t,x) + \Lambda^+\tilde{u}_x(t,x) &= \Sigma^{++}\tilde{u}(t,x) + \Sigma^{+-}\tilde{v}(t,x)\\ &\quad - P^+(x)\tilde{v}(t,0) \qquad (68)\\ \tilde{v}_t(t,x) + \Lambda^-\tilde{v}_x(t,x) &= \Sigma^{-+}\tilde{u}(t,x) + \Sigma^{--}\tilde{v}(t,x)\\ &\quad - P^-(x)\tilde{v}(t,0) \qquad (69)\end{aligned}$$

with the boundary conditions

$$\tilde{u}(t,0) = 0, \qquad \tilde{v}(t,1) = R_1\tilde{u}(t,1). \tag{70}$$

6.2. Target system

We map the system (68)–(70) to the following system

$$\begin{aligned}\tilde{\alpha}_t(t,x) + \Lambda^+\tilde{\alpha}_x(t,x) &= \Sigma^{++}\tilde{\alpha}(t,x)\\ &\quad + \int_0^x D^+(x,\xi)\tilde{\alpha}(t,\xi)d\xi \qquad (71)\\ \tilde{\beta}_t(t,x) - \Lambda^-\tilde{\beta}_x(t,x) &= \Sigma^{-+}\tilde{\alpha}(t,x) + \Omega(x)\beta(t,x)\\ &\quad + \int_0^x D^-(x,\xi)\tilde{\alpha}(t,\xi)d\xi \qquad (72)\end{aligned}$$

with the following boundary conditions

$$\tilde{\alpha}(t,0) = 0, \qquad \tilde{\beta}(t,1) = R_1\tilde{\alpha}(t,1) \tag{73}$$

where D^+, and D^- are L^∞ matrix functions of the domain $\mathcal{T}$ and $\tilde{\Omega} \in L^\infty(0,1)$ is an upper triangular matrix with the following structure

$$\tilde{\Omega}(x) = \begin{pmatrix}\tilde{\omega}_{1,1}(x) & \tilde{\omega}_{1,2}(x) & \cdots & \tilde{\omega}_{1,m}(x)\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \tilde{\omega}_{m-1,m-1}(x) & \tilde{\omega}_{m-1,m}(x)\\ 0 & \cdots & 0 & \tilde{\omega}_{m,m}(x)\end{pmatrix}. \tag{74}$$

Lemma 3. *The system (71), (72) reaches its zero equilibrium in a finite time t_F where t_F is defined by (10).*

Proof. The proof similar to that of Lemma 1 is omitted.

6.3. Volterra transformation

In order to map the original system (68)–(70) to the target system (71)–(73), we use the following Volterra transformation

$$\tilde{u}(t,x) = \tilde{\alpha}(t,x) + \int_0^x M(x,\xi)\tilde{\beta}(t,\xi)d\xi \tag{75}$$

$$\tilde{v}(t,x) = \tilde{\beta}(t,x) + \int_0^x N(x,\xi)\tilde{\beta}(t,\xi)d\xi \tag{76}$$

where the kernels M and N defined on $\mathcal{T} = \{(x,\xi) \in [0,1]^2 | \xi \le x\}$ have yet to be defined. Differentiating (75), (76) with respect to space and time yields the following kernel equations

for $1 \le i \le n$, $1 \le j \le m$

$$\begin{aligned}\lambda_i\partial_x M_{ij}(x,\xi) - \mu_j\partial_\xi M_{ij}(x,\xi) &= \sum_{k=1}^n \sigma_{ik}^{++}M_{kj}(x,\xi)\\ &\quad + \sum_{p=1}^m \sigma_{ip}^{-+}N_{pj}(x,\xi) - \sum_{p=1}^m M_{ip}(x,\xi)\tilde{\omega}_{pj}(x) \qquad (77)\end{aligned}$$

for $1 \le i \le m$, $1 \le j \le m$

$$\begin{aligned}\mu_i\partial_x N_{ij}(x,\xi) + \mu_j\partial_\xi N_{ij}(x,\xi) &= -\sum_{k=1}^n \sigma_{ik}^{--}N_{kj}(x,\xi)\\ &\quad - \sum_{p=1}^n \sigma_{ip}^{+-}M_{pj}(x,\xi) + \sum_{p=1}^m N_{ip}(x,\xi)\tilde{\omega}_{pj}(x) \qquad (78)\end{aligned}$$

with the following set of boundary conditions:

$$\forall 1 \le i \le m,\ \forall j \le n, \quad M_{ij}(x,x) = -\frac{\sigma_{ij}^{+-}}{\mu_j + \lambda_i} = k_{ij} \tag{79}$$

$$\forall 1 \le i,j \le m,\ j < i \quad N_{ij}(x,x) = \frac{-\sigma_{ij}^{--}}{\mu_j - \mu_i} \tag{80}$$

$$\forall i \le j \quad \tilde{\omega}_{ij}(x) = (\mu_j - \mu_i)N_{ij}(x,x) + \sigma_{ij}^{--}. \tag{81}$$

Evaluating (75), (76) at $x = 1$ yields

$$\forall 1 \le i,j \le m, \quad N_{ij}(1,\xi) = \sum_{k=1}^n \rho_{ik}M_{kj}(1,\xi) \tag{82}$$

while d_{ij}^+, d_{ij}^- are given by

$$\begin{aligned}d_{ij}^+(x,\xi) &= -\sum_{k=1}^m M_{ik}(x,\xi)\sigma_{kj}^{-+}\\ &\quad + \int_\xi^x \sum_{k=1}^m M_{ik}(x,s)d_{kj}^-(s,\xi)ds \qquad (83)\end{aligned}$$

$$d_{ij}^{-}(x,\xi) = -\sum_{k=1}^{m} N_{ik}(x,\xi)\sigma_{kj}^{-+} + \int_{\xi}^{x}\sum_{k=1}^{m} N_{ik}(x,s)d_{kj}^{-}(s,\xi)ds \tag{84}$$

provided the M and N kernels are well-defined. Finally the observer gains are given by

$$p_{ij}^{+}(x) = \mu_j M_{ij}(x,0) \tag{85}$$

$$p_{ij}^{-}(x) = \mu_j N_{ij}(x,0). \tag{86}$$

Considering the following alternate variables

$$\bar{M}_{ij}(\chi,y) = M_{ij}(1-y,1-\chi) = M_{ij}(x,\xi) \tag{87}$$

$$\bar{N}_{ij}(\chi,y) = N_{ij}(1-y,1-\chi) = N_{ij}(x,\xi) \tag{88}$$

$$\bar{\omega}_{ij}(\chi) = \tilde{\omega}_{ij}(x) \tag{89}$$

one can prove that this system has the same cascade structure as the controller kernel system. Using a similar proof we can assess its well-posedness.

6.4. Output feedback controller

The estimates can be used in an observer–controller to derive an output feedback law yielding finite-time stability of the zero equilibrium.

Lemma 4. *Consider the system composed of (1)–(3) and target system (65)–(67) with the following control law*

$$U(t) = \int_0^1 [K(1,\xi)\hat{u}(t,\xi) + L(1,\xi)\hat{v}(t,\xi)]d\xi - R_1\hat{u}(t,1) \tag{90}$$

where K and L are defined by (27)–(32). Its solutions $(u, v, \hat{u}, \hat{v})$ converge in finite time to zero.

Proof. The convergence of the observer error states $\tilde{u}, \tilde{v}$ to zero for $t_F \le t$ is ensured by Lemma 3, along with the existence of the backstepping transformation. Thus, once $t_F \le t$, $v(t,0) = \hat{v}(t,0)$ and one can use Theorem 2. Therefore for $2t_F \le t$, one has $(\tilde{u}, \tilde{v}, \hat{u}, \hat{v}) \equiv 0$ which yields $(u, v) \equiv 0$.

7. Simulation results

In this section we illustrate our results with simulations on a toy problem. The algorithm we use follows the proof of Theorem 1. It solves the integral equations (42)–(45) computing the characteristic lines and finding the fixed point of Φ defined by (55), solution of the kernel equations. These kernels are then used to compute the control law. The numerical values of the parameters are as follows.

$$n = 1, \quad m = 2, \quad \lambda_1 = 10, \quad \mu_1 = 4, \quad \mu_2 = 5 \tag{91}$$

$$\Sigma^{++} = 1 \quad \Sigma^{+-} = \begin{pmatrix} 4 & 4 \end{pmatrix} \quad \Sigma^{-+} = \begin{pmatrix} 10 \\ 10 \end{pmatrix} \tag{92}$$

$$\Sigma^{--} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \quad Q_0 = 0 \quad R_1 = 0. \tag{93}$$

The parameters values are chosen such that

- the system is strongly unstable (Bastin & Coron, 2015), in particular not stabilizable by a static feedback law, and
- there is a large benefit in using the presented result compared to (Hu et al., 2015) since the minimum time $t_{F_1} = \frac{1}{\lambda_1} + \frac{1}{\mu_1} = 0.35$ is almost half of $t_{F_2} = \frac{1}{\lambda_1} + \frac{1}{\mu_1} + \frac{1}{\mu_2} = 0.55$.

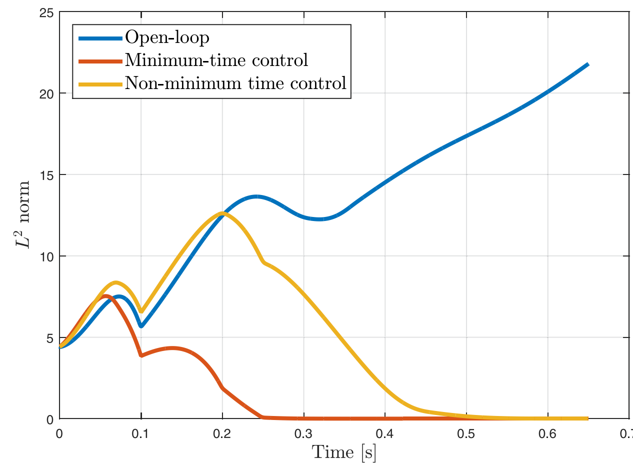

Fig. 1. Time evolution of the L^2-norm in open loop and using two different controllers.

Intuitively, these two points would suggest that the control effort required to achieve minimum time convergence be greater than the slower one for the chosen parameters. This intuition is further reinforced by (38) and (36) that respectively suggest that

- close transport velocities yield larger control gains
- the magnitude of the gains increases with the number of leftward convective states[1] contrary to Hu et al. (2015).

However, as it appears the simulation results are contrary to the intuition. Fig. 1 pictures the $\mathcal{L}^2$-norm of the state (u, v) in open loop, using the control law presented in Hu et al. (2015) and then using the control law (61) presented in this paper. The convergence times are consistent with the theory. Fig. 2 depicts the total control effort $\bar{U}(t)$ defined by $\bar{U}(t) = U_1^2(t) + U_2^2(t)$ which is significantly lower for the minimum-time control. This surprising result may be explained as follows: the control gains depicted in absolute value on Fig. 3 are of comparable size on a large part of the spatial domain. Since the control law takes the form of a spatial integral, the two controllers are expected to yield similar magnitude of control action for a given norm of the states. The non-minimum time control "waits" for fast states to converge before stabilizing slower states, which exponentially grows in the mean time. This result combined with a larger overshoot (as depicted on Fig. 2) entails a larger control effort.

8. Concluding remarks

Using the backstepping approach we have presented a stabilizing boundary feedback law for a general class of linear first-order systems. Moreover, contrary to Hu et al. (2015), the zero-equilibrium of the system is reached in minimum time t_F.

The presented design raises several important questions that will be the topic of future investigation. In Hu et al. (2015), the proposed control law does not yield minimum time convergence, but features several degrees of freedom that may be usable to handle transients. A comparison of the control effort, of the transient responses of both designs, as well as their comparative robustness, should be performed.

Besides, the presented result narrows the gap with the theoretical controllability results of Li and Rao (2010). These results, although they do not provide explicit control law, ensure exact minimum-time controllability with less control inputs than

[1] Due to the recursive dependence of the kernels from one line to the next.

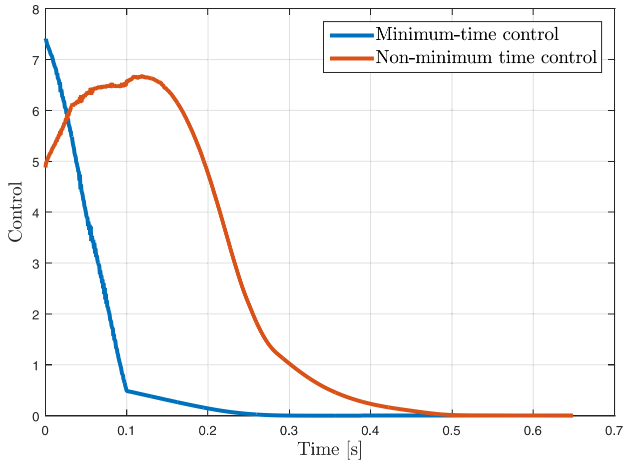

Fig. 2. Controller dynamics.

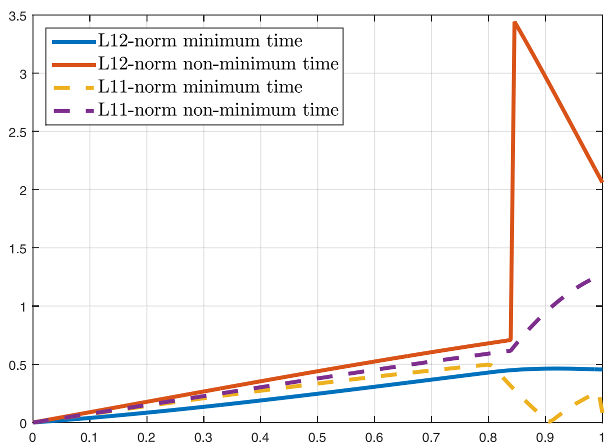

Fig. 3. Controller gains.

what is currently achievable using backstepping. More generally, this raises the question of the links between controllability and stabilization by backstepping. In particular future works will consider first-order systems with controls in each boundary. In this case, the current method cannot straightforwardly be applied.

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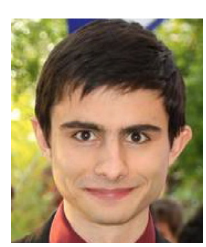

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