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Estimating the Left Boundary Condition of Coupled 1-D Linear Hyperbolic PDEs From Right Boundary Sensing

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***Abstract*— In this paper, we develop an adaptive observer for $n+1$ coupled first-order 1-D hyperbolic PDEs for estimation of unknown parameters appearing in the left boundary condition using only measurements at the right boundary. Proof of boundedness of the estimated parameters and sufficient conditions for convergence are offered and the result is demonstrated in a simulation.**

I. Introduction

Many physical systems can be modeled using first order hyperbolic linear partial differential equations (PDEs). Representative engineering applications such as heat exchangers [1], transmission lines [2], oil wells [3], road traffic [4] and multiphase flow [5], to mention a few, involve convection phenomena with spatio-temporal dynamics. Due to the wide area of applications, such systems have been subject to extensive research during the last decades, and we refer to [6], [7], [8] for some significant control related results.

The backstepping method from nonlinear control theory was recently extended to PDEs. The key point of this approach is the introduction of an invertible Volterra transformation that maps the original system of PDEs into a simpler target system whose stability is easier to establish. The invertibility of the transformation allows to state the equivalence of stability properties for the two systems. The backstepping method applied to PDEs was initially developed for PDEs of the parabolic type in [9], and has later been adopted to first order hyperbolic systems [10]. In [11], it was extended to two coupled first order systems, with the general $n+1$ case derived in [12]. For such systems, n PDEs convect in one direction, with a single PDE convecting in the opposite direction. The stabilization result proposed in [12] was extended even further to general $n+m$ systems in [13], with an arbitrary number of PDEs in each direction and both controllers and observers using boundary sensing only have been developed.

Adaptive control using backstepping was investigated for parabolic PDEs in [14], where a certainty equivalence based backstepping scheme was used. Adaptive backstepping for hyperbolic PDEs, however, is relatively new. An early result seems to be [15], where a hyperbolic partial integro-differential equation was adaptively stabilized using boundary sensing only. Later, an adaptive observer for estimating additive disturbance terms in the boundary condition of hyperbolic systems was investigated in [16]. The derived method was applied to a problem from underbalanced drilling. In [17], backstepping was used in conjunction with sliding mode control to design an adaptive controller estimating and taking into account an uncertain parameter in the boundary condition at the same boundary as actuation. We also mention that the observers designed for the disturbance rejection problem in [18], [19] and [20], and for the leak detection problem in [21] can be interpreted as adaptive observers for hyperbolic PDEs. Very recently, adaptive observers for estimating both additive and multiplicative unknown parameters in the boundary conditions was derived by both [22] and [23]. The former was based on swapping design, while the latter was based on a Lyapunov approach and backstepping. However, both these methods require sensing at both boundaries.

In this paper, we seek to estimate both unknown multiplicative and additive parameters in the left boundary condition of $n + 1$ coupled linear hyperbolic PDEs, with sensing limited to the right boundary. We will combine the swapping design-based observer from [22] with the backstepping transformation in [23] to achieve this. The motivation for this problem formulation comes from oil wells, whose multiphase flow dynamics is governed by hyperbolic PDEs, where uncertain parameters typically appear in the downhole boundary condition, while measurements are limited to the topside platform.

This paper is organized as follows: in Section II we present the dynamic model and pose the estimation problem. In Section III, we design a set of filters that provide a static relationship between the system states and the unknown parameters in the boundary condition, with an error that is shown to converge to zero in finite time in Section IV. The filters rely on a particular signal on the left boundary, which is not measured, but in Section V it is constructed in closed form using only available measurements. An update law with a proof of boundedness of the estimated parameters is presented in Section VI, and some concluding remarks are given in Section VIII.

II. Problem statement

We investigate systems that can be used to model multiphase flow, stated as

$$u_t(x,t) + \Lambda u_x(x,t) = \Sigma u(x,t) + \omega v(x,t) \tag{1}$$

$$v_t(x,t) - \mu v_x(x,t) = \theta^T u(x,t) + \pi v(x,t) \tag{2}$$

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with boundary conditions

$$u(0,t) = qv(0,t) + d \tag{3}$$
$$v(1,t) = \rho^T u(1,t) + U(t) \tag{4}$$

where,

$$u(x,t) = \begin{bmatrix} u_1(x,t) & \dots & u_n(x,t) \end{bmatrix}^T \tag{5}$$

$$\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}, \quad \rho = \begin{bmatrix} \rho_1 & \dots & \rho_n \end{bmatrix}^T. \tag{6}$$

$$\Sigma = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nn} \end{bmatrix}, \omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}, \theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix} \tag{7}$$

and

$$-\mu < 0 < \lambda_1 < \lambda_2 < \dots < \lambda_n. \tag{8}$$

The initial conditions satisfy

$$v^0(x),\ u_i^0(x) \in \mathcal{L}^2([0,1]),\ i = 1 \dots n. \tag{9}$$

The aim is to estimate the boundary parameters $q = \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix}^T$ and $d = \begin{bmatrix} d_1 & \dots & d_n \end{bmatrix}^T$ from sensing at $x = 1$ only, that is, the only available measurement is

$$y(t) = u(1,t). \tag{10}$$

The term $U(t)$ can be considered a control input, although closed loop control is not investigated in this paper.

While the method extends to spatially varying coefficients in (1)–(2), we consider here constant coefficients for the sake of readability.

III. Filter design

By defining a set of filters, we will show that a time-delayed version of the system states can be expressed as a linear combination of the filters and the unknown parameters q and d.

A. Output filter

First off, we will need to store past values of the measurement $y(t)$. Introducing

$$\mathcal{Y}(t) = \begin{bmatrix} \mathcal{Y}_1(t) & \mathcal{Y}_2(t) & \dots & \mathcal{Y}_n(t) \end{bmatrix}^T \tag{11}$$

the filter

$$\mathcal{Y}_t(x,t) + \lambda \mathcal{Y}_x(x,t) = 0 \tag{12}$$

with boundary condition

$$\mathcal{Y}(0,t) = y(t) \tag{13}$$

where

$$\lambda := \min_i \lambda_i = \lambda_1 \tag{14}$$

provides past values of $y(t)$, since

$$y(t - \lambda^{-1}x) = \mathcal{Y}(x,t). \tag{15}$$

B. Input filters

We introduce the "input filters"

$$a_t(x,t) + \Lambda a_x(x,t) = \Sigma a(x,t) + \omega b(x,t) - K_1(x)(y(t) - a(1,t)) \tag{16}$$
$$b_t(x,t) - \mu b_x(x,t) = \theta^T a(x,t) + \pi b(x,t) - K_2(x)(y(t) - a(1,t)) \tag{17}$$

with boundary conditions

$$a(0,t) = 0 \tag{18}$$
$$b(1,t) = \rho^T y(t) + U(t). \tag{19}$$

where

$$a(x,t) = \begin{bmatrix} a_1(x,t) & \dots & a_n(x,t) \end{bmatrix}^T \tag{20}$$

and the injection gains $K_1(x)$ and $K_2(x)$ are to be designed later. These filters model how the input $\rho^T y(t) + U(t)$ affect the system states u and v. We also construct the time-delay filters

$$\mathcal{A}_t(x,\xi,t) + \lambda \mathcal{A}_x(x,\xi,t) = 0 \tag{21}$$
$$\mathcal{B}_t(x,\xi,t) + \lambda \mathcal{B}_x(x,\xi,t) = 0 \tag{22}$$

with boundary conditions

$$\mathcal{A}(0,\xi,t) = a(\xi,t) \tag{23}$$
$$\mathcal{B}(0,\xi,t) = b(\xi,t). \tag{24}$$

where

$$\mathcal{A}(x,\xi,t) = \begin{bmatrix} \mathcal{A}_1(x,\xi,t) & \dots & \mathcal{A}_n(x,\xi,t) \end{bmatrix}^T. \tag{25}$$

The latter filters make sure previous values of a and b are available.

C. Parameter filters

Next, we design parameter filters that are used to model how the parameters q and d affect the system states u and v. Define

$$P_t(x,t) + \Lambda P_x(x,t) = \Sigma P(x,t) + \omega r^T(x,t) + K_1(x)P(1,t) \tag{26}$$
$$r_t^T(x,t) - \mu r_x^T(x,t) = \theta^T P(x,t) + \pi r^T(x,t) + K_2(x)P(1,t) \tag{27}$$

with boundary conditions

$$P(0,t) = I_{n\times n} \cdot \bar{p}(t) \tag{28}$$
$$r^T(1,t) = 0 \tag{29}$$

where

$$P(x,t) = \{p_{ij}(x,t)\}_{1\le i,j \le n} \tag{30}$$
$$r^T(x,t) = \begin{bmatrix} r_1(x,t) & r_2(x,t) & \dots & r_n(x,t) \end{bmatrix} \tag{31}$$

and the signal $\bar{p}(t)$ is sought to be designed so that

$$\bar{p}(t) = v(0, t - \lambda^{-1}), \tag{32}$$

but from available measurements and filters only. The actual $\bar{p}(t)$ is designed in Section V, but for now we assume that (32) is the case. Lastly, define

$$\begin{aligned} W_t(x,t) + \Lambda W_x(x,t) &= \Sigma W(x,t) + \omega z^T(x,t) \\ &\quad + K_1(x)W(1,t) \qquad (33) \\ z_t^T(x,t) - \mu z_x^T(x,t) &= \theta^T W(x,t) + \pi z^T(x,t) \\ &\quad + K_2(x)W(1,t) \qquad (34) \end{aligned}$$

with boundary conditions

$$\begin{aligned} W(0,t) &= I_{n\times n} \qquad (35) \\ z^T(1,t) &= 0 \qquad (36) \end{aligned}$$

where

$$\begin{aligned} W(x,t) &= \{w_{ij}(x,t)\}_{1\le i,j\le n} \qquad (37) \\ z^T(x,t) &= \begin{bmatrix} z_1(x,t) & z_2(x,t) & \dots & z_n(x,t) \end{bmatrix} \qquad (38) \end{aligned}$$

D. Relationship to the system states

With the filters derived above, we find the following relations to the system states

$$\begin{aligned} u(x,t-\lambda^{-1}) &= \mathcal{A}(1,x,t) + P(x,t)q + W(x,t)d \\ &\quad + e(x,t) \qquad (39) \\ v(x,t-\lambda^{-1}) &= \mathcal{B}(1,x,t) + r^T(x,t)q + z^T(x,t)d \\ &\quad + \epsilon(x,t) \qquad (40) \end{aligned}$$

with the error terms having the following dynamics

$$\begin{aligned} e_t(x,t) + \Lambda e_x(x,t) &= \Sigma e(x,t) + \omega\epsilon(x,t) \\ &\quad + K_1(x)e(1,t) \qquad (41) \\ \epsilon_t(x,t) - \mu\epsilon_x(x,t) &= \theta^T e(x,t) + \pi\epsilon(x,t) \\ &\quad + K_2(x)e(1,t) \qquad (42) \end{aligned}$$

with boundary conditions

$$\begin{aligned} e(0,t) &= 0 \qquad (43) \\ \epsilon(1,t) &= 0. \qquad (44) \end{aligned}$$

IV. Error dynamics analysis

We will here show that the errors $e(x,t)$ and $\epsilon(x,t)$ in (39)–(40) go to zero in finite time by a clever choice of the injection terms $K_1(x)$ and $K_2(x)$. We proceed by performing a backstepping transformation on the error system (41)–(42). The system has the same form as the error dynamics for the non-adaptive observer presented in [23] where they used the following backstepping transformation

$$e(x,t) = \alpha(x,t) + \int_x^1 M(x,\xi)\alpha(\xi,t)d\xi \qquad (45)$$

$$\epsilon(x,t) = \beta(x,t) + \int_x^1 N(x,\xi)\alpha(\xi,t)d\xi \qquad (46)$$

with

$$\begin{aligned} K_1(x) &= -M(x,1)\Lambda \qquad (47) \\ K_2(x) &= -N(x,1)\Lambda \qquad (48) \end{aligned}$$

transforming the following target error system

$$\begin{aligned} \alpha_t(x,t) + \Lambda\alpha_x(x,t) &= \bar{\Sigma}\alpha(x,t) \\ &\quad + \omega\beta(x,t) - \int_x^1 D^+(x,\xi)\beta(\xi,t)d\xi \qquad (49) \\ \beta_t(x,t) - \mu\beta_x(x,t) &= \pi\beta(x,t) \\ &\quad - \int_x^1 d^-(x,\xi)\beta(\xi,t)d\xi \qquad (50) \end{aligned}$$

with boundary conditions

$$\begin{aligned} \alpha(0,t) &= \int_0^1 H(\xi)\alpha(\xi,t)d\xi \qquad (51) \\ \beta(1,t) &= 0 \qquad (52) \end{aligned}$$

to the original system (41)–(44), where $\bar{\Sigma}$ is a diagonal matrix on the form

$$\bar{\Sigma} = \mathrm{diag}\{\sigma_{11}, \sigma_{22}, \dots, \sigma_{nn}\} \qquad (53)$$

and

$$D^+(x,\xi) = M(x,\xi)\omega + \int_\xi^x M(x,\eta)D^+(\eta,\xi)d\eta \qquad (54)$$

$$d^-(x,\xi) = N(x,\xi)\omega + \int_\xi^x N(x,\eta)D^+(\eta,\xi)d\eta \qquad (55)$$

with $H(x) = \{h_{ij}(x)\}_{1\le i,j\le n}$ as a strict lower triangular matrix with components

$$\begin{aligned} h_{ij}(x) &= -M_{ij}(0,\xi), \text{ for } 1\le j<i\le n \qquad (56) \\ h_{ij}(x) &= 0, \text{ otherwise.} \qquad (57) \end{aligned}$$

The kernels

$$\begin{aligned} M(x,\xi) &= \{m_{ij}(x,\xi)\}_{1\le i,j\le n} \qquad (58) \\ N(x,\xi) &= \begin{bmatrix} n_1(x,\xi) & n_2(x,\xi) & \dots & n_n(x,\xi) \end{bmatrix} \qquad (59) \end{aligned}$$

in the backstepping transformation (45)–(46) are defined over the triangular domain $\mathcal{T} = \{(x,\xi) \mid 0\le x\le\xi\le 1\}$, and satisfy the following PDEs, for $1\le i,j\le n$

$$\begin{aligned} &\lambda_i\partial_x m_{ij}(x,\xi) + \lambda_j\partial_\xi m_{ij}(x,\xi) \\ &\quad = \sum_{k=1}^n \sigma_{ik}m_{kj}(x,\xi) + \omega_i n_j(x,\xi) - \sigma_{jj}m_{ij}(x,\xi) \qquad (60) \end{aligned}$$

and for $1\le i\le n$

$$\begin{aligned} &-\mu\partial_x n_i(x,\xi) + \lambda_i\partial_\xi(x,\xi) \\ &\quad = \sum_{k=1}^n \theta_k m_{ki}(x,\xi) + \pi n_i(x,\xi) - \sigma_{ii}n_i(x,\xi) \qquad (61) \end{aligned}$$

with the following boundary conditions

$$\begin{aligned} m_{ij}(x,x) &= \frac{\sigma_{ij}}{\lambda_j-\lambda_i}, \text{ for } 1\le i,j\le n, \quad i\ne j \qquad (62) \\ n_i(x,x) &= \frac{\theta_i}{\mu_i+\lambda_i}, \text{ for } 1\le i\le n \qquad (63) \\ m_{ij}(0,\xi) &= 0, \qquad \text{ for } 1\le i\le j\le n \qquad (64) \\ m_{ij}(x,1) &= \frac{\sigma_{ij}}{\lambda_j-\lambda_i}, \text{ for } 1\le j<i\le n \qquad (65) \end{aligned}$$

It was shown in [13] that the transformation (45)–(46) is invertible, that the kernel equation (60)–(65) has a unique solution and that the target system reaches zero in a finite time given by

$$t_F = \mu^{-1} + \sum_{k=1}^{n} \lambda_k^{-1}. \tag{66}$$

The fact that convergence to zero is achieved in finite time comes from the cascaded property of the system (49)–(52), with initially only β being independent of the α. After the time μ^{-1}, α_1 is independent of the other signals and will be zero after an additional time λ^{-1}. Then, α_2 is independent of the remaining signals and so on. Since the transformation (45)–(46) is invertible, the stability properties of the target and original system are equivalent.

V. Design of $\bar{p}(t)$

The filters designed in Section III require (32), but $v(0,t)$ is not measured. It turns out, however, that $v(0, t-\lambda^{-1})$ can be expressed in closed form using available measurements, only. Construct the signals

$$\tilde{u} = u - a \tag{67}$$

$$\tilde{v} = v - b \tag{68}$$

which have the following dynamics;

$$\tilde{u}_t(x,t) + \Lambda \tilde{u}_x(x,t) = \Sigma \tilde{u}(x,t) + \omega \tilde{v}(x,t) + K_1(x)\tilde{u}(1,t) \tag{69}$$

$$\tilde{v}_t(x,t) - \mu \tilde{v}_x(x,t) = \theta^T \tilde{u}(x,t) + \pi \tilde{v}(x,t) + K_2(x)\tilde{u}(1,t) \tag{70}$$

with the boundary conditions

$$\tilde{u}(0,t) = qv(0,t) + d \tag{71}$$

$$\tilde{v}(1,t) = 0 \tag{72}$$

We use the same type of backstepping transformation as in the previous section; let

$$\tilde{\alpha}(x,t) = \tilde{u}(x,t) - \int_x^1 M(x,\xi)\tilde{\alpha}(\xi,t)d\xi \tag{73}$$

$$\tilde{\beta}(x,t) = \tilde{v}(x,t) - \int_x^1 N(x,\xi)\tilde{\alpha}(\xi,t)d\xi \tag{74}$$

to produce

$$\tilde{\alpha}_t(x,t) + \Lambda \tilde{\alpha}_x(x,t) = \bar{\Sigma}\tilde{\alpha}(x,t) + \omega \tilde{\beta}(x,t) - \int_x^1 D^+(x,\xi)\tilde{\beta}(\xi,t)d\xi \tag{75}$$

$$\tilde{\beta}_t(x,t) - \mu \tilde{\beta}_x(x,t) = \pi \tilde{\beta}(x,t) - \int_x^1 d^-(x,\xi)\tilde{\beta}(\xi,t)d\xi \tag{76}$$

with boundary conditions

$$\tilde{\alpha}(0,t) = \int_0^1 H(\xi)\tilde{\alpha}(\xi,t)d\xi + qv(0,t) + d \tag{77}$$

$$\tilde{\beta}(1,t) = 0 \tag{78}$$

where $\bar{\Sigma}$, $M(x,\xi)$, $N(x,\xi)$, $D^+(x,\xi)$, $d^-(x,\xi)$ and $H(\xi)$ are as in the previous section. From the structure of the system consisting of (76) and (78), we will after $t = t_0 = \mu^{-1}$, have $\tilde{\beta} \equiv 0$[1], and the target error system reduces to

$$\tilde{\alpha}_t(x,t) = -\Lambda \tilde{\alpha}_x(x,t) + \bar{\Sigma}\tilde{\alpha}(x,t) \tag{79}$$

with boundary condition

$$\tilde{\alpha}(0,t) = \int_0^1 H(\xi)\tilde{\alpha}(\xi,t)d\xi + qv(0,t) + d \tag{80}$$

or when written out, yields a set of uncoupled PDEs

$$\partial_t \tilde{\alpha}_i(x,t) + \lambda_i \partial_x \tilde{\alpha}_i(x,t) = \sigma_{ii}\tilde{\alpha}_i(x,t) \tag{81}$$

with boundary conditions

$$\tilde{\alpha}_i(0,t) = \sum_{j=1}^{i-1} \int_0^1 h_{ij}(\xi)\tilde{\alpha}_j(\xi,t)d\xi + q_i v(0,t) + d_i. \tag{82}$$

The solution of (81) is

$$\tilde{\alpha}_i(x,t) = \exp\left(\frac{\sigma_{ii}}{\lambda_i}x\right)\tilde{\alpha}_i(0, t-\lambda^{-1}x). \tag{83}$$

Particularly, we find

$$\tilde{\alpha}_i(1,t) = u_i(1,t) - a_i(1,t) = y_i(t) - a_i(1,t) \tag{84}$$

which means that $\tilde{\alpha}_i(x,t)$ can be expressed using $y_i(t)$ and $a_i(1,t)$ as follows

$$\tilde{\alpha}_i(x,t) = \exp\left(-\frac{\sigma_{ii}}{\lambda_i}(1-x)\right) \times \left[y_i(t+\lambda_i^{-1}(1-x)) - a_i(1, t+\lambda_i^{-1}(1-x))\right] \tag{85}$$

Again using the fact that $\tilde{\beta} \equiv 0$ for $t \geq t_0 = \mu^{-1}$, we specifically have that $\tilde{\beta}(0,t) = 0$, and from (74) we find

$$v(0,t) = b(0,t) + \int_0^1 N(0,\xi)\tilde{\alpha}(\xi,t)d\xi. \tag{86}$$

Using (85), we may write

$$v(0,t) = b(0,t) + \sum_{i=1}^{n} \int_0^1 n_i(0,\xi)\exp\left(-\frac{\sigma_{ii}}{\lambda_i}(1-\xi)\right) \times \left[y_i(t+\lambda_i^{-1}(1-\xi)) - a_i(1,t+\lambda_i^{-1}(1-\xi))\right] d\xi \tag{87}$$

where $n_i(0,\xi)$ are the components of $N(0,\xi)$. This yields a way to compute $v(0,t)$ from the measurements and the signal $b(0,t)$. However, it depends upon future values of $y(t)$ and $a_i(1,t)$. What may be computed, however, is a time-delayed $v(0,t)$

$$v(0,t-\lambda^{-1}) = \mathcal{B}(1,0,t) + \sum_{i=1}^{n} \int_0^1 n_i(0,\xi)\exp\left(-\frac{\sigma_{ii}}{\lambda_i}(1-\xi)\right) \times \left[\mathcal{Y}_i(1-\lambda_i^{-1}\lambda(1-\xi),t) - \mathcal{A}_i(1-\lambda_i^{-1}\lambda(1-\xi),1,t)\right] d\xi. \tag{88}$$

[1]This can be observed from noting that the integral source term only involves points ξ over $x \leq \xi \leq 1$.

Thus, we choose

$$\bar{p}(t) = \mathcal{B}(1,0,t) + \sum_{i=1}^{n} \int_0^1 n_i(0,\xi) \exp\left(-\frac{\sigma_{ii}}{\lambda_i}(1-\xi)\right) \times \left[\mathcal{Y}_i(1-\lambda_i^{-1}\lambda(1-\xi),t) - \mathcal{A}_i(1-\lambda_i^{-1}\lambda(1-\xi),1,t)\right] d\xi \tag{89}$$

which makes $\bar{p}(t)$ equal $v(0,t-\lambda^{-1})$ and expressed using only available signals.

VI. Update law

From the static form (39)–(40) with error terms converging to zero, one can follow the same approach as in [22] and use standard gradient and least squares update laws to estimate the unknown parameters in q and d. From (39), we have

$$\begin{aligned} e(1,t) &= u(1,t-\lambda^{-1}) - a(1,t-\lambda^{-1}) - P(1,t)q \\ &\quad - W(1,t)d \\ &= \mathcal{Y}(1,t) - \mathcal{A}(1,1,t) - P(1,t)q - W(1,t)d \end{aligned} \tag{90}$$

Define the following vector

$$h(t) := \mathcal{Y}(1,t) - \mathcal{A}(1,1,t) \tag{91}$$

and the matrix

$$R(t) = \begin{bmatrix} P(1,t) & W(1,t) \end{bmatrix} \tag{92}$$

then

$$e(1,t) = h(t) - R(t)\nu. \tag{93}$$

where

$$\nu = \begin{bmatrix} q^T & d^T \end{bmatrix}^T. \tag{94}$$

Theorem 1 (Modified Theorem 2 in [22]): Consider the system (1)–(4) with filters (16)–(19), (21)–(24), (26)–(29), (33)–(36) with $\bar{p}(t)$ given by (89) and with injection gains given by (47)–(48). Then the following normalized update law

$$\dot{\hat{\nu}} = s(t)\Gamma \frac{R^T(t)(h(t) - R(t)\hat{\nu})}{1 + ||R^T(t)R(t)||^2} \tag{95}$$

where

$$s(t) = \begin{cases} 1 & \text{if } t > t_F \\ 0 & \text{otherwise} \end{cases} \tag{96}$$

with t_F given in (66), ensures that $\tilde{\nu} = \hat{\nu} - \nu \in \mathcal{L}_\infty$. Moreover, if $R(t)$ and $\dot{R}(t)$ are bounded and $R^T(t)$ is persistently exciting (PE), that is; there exists positive constants T_0, c_0, c_1 so that

$$c_1 I_{2n\times 2n} \geq \frac{1}{T_0}\int_t^{t+T_0} R^T(\tau)R(\tau)d\tau \geq c_0 I_{2n\times 2n} \tag{97}$$

for all $t \geq 0$, then the system parameters converge to their true values exponentially.

Proof: We construct a "prediction error" as follows

$$\hat{e}(1,t) = h(t) - R(t)\hat{\nu}. \tag{98}$$

Now consider the Lyapunov function candidate

$$V = \frac{1}{2}\tilde{\nu}^T\Gamma^{-1}\tilde{\nu} \tag{99}$$

where $\tilde{\nu} := \hat{\nu} - \nu$. Then

$$\dot{V} = \tilde{\nu}^T\Gamma^{-1}\dot{\tilde{\nu}} = s(t)\tilde{\nu}^T \frac{R^T(t)\hat{e}(1,t)}{1 + ||R^T(t)R(t)||^2} \tag{100}$$

Noticing that $\hat{e}(1,t) = h(t) - R(t)\hat{\nu} = e(1,t) - R(t)\tilde{\nu}$, we find

$$\dot{V} = s(t)\frac{\tilde{\nu}^T R^T(t)e(1,t)}{1 + ||R^T(t)R(t)||^2} - s(t)\frac{|R(t)\tilde{\nu}|^2}{1 + ||R^T(t)R(t)||^2}. \tag{101}$$

The first term is zero, since, for $t \leq t_F$, we have $s(t) = 0$, and for $t > t_F$, we have $e(1,t) = 0$. Hence

$$\dot{V} = -s(t)\frac{|R(t)\tilde{\nu}|^2}{1 + ||R^T(t)R(t)||^2}. \tag{102}$$

Which shows that V is non-increasing and hence bounded, which in turn implies that $\tilde{\nu}$ is bounded.

Since, from Section IV, we have that e and ϵ will go to zero in a finite time and independently of the update law of Theorem 1, the static form of the measurements in (93) eventually reach the form $h(t) = R(t)\nu$. The latter part of the theorem then follows from part iii) of Theorem 4.3.2 in [24]. ∎

Remark 2: A stable plant will ensure that $R(t)$ and $\dot{R}(t)$ are bounded.

VII. Simulation

The system and the adaptive observer were implemented in Matlab for $n = 2$. The system transport speeds were set to

$$\lambda_1 = \mu = 1, \quad \lambda_2 = 2 \tag{103}$$

with the in-domain parameters set to

$$\begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & \omega_1 \\ \sigma_{2,1} & \sigma_{2,2} & \omega_2 \\ \theta_1 & \theta_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.83 & 0.07 \\ -0.4 & 0 & -0.2 \\ -0.9 & 0.5 & 0 \end{bmatrix} \tag{104}$$

and the boundary parameters set to

$$\begin{bmatrix} q_1 & d_1 & \rho_1 \\ q_2 & d_2 & \rho_2 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 & 0.1 \\ 0.2 & -0.4 & 0.1 \end{bmatrix}. \tag{105}$$

This plant is open-loop stable. The initial values for the plant were set to

$$u_1(x,0) = \sin(x),\ u_2(x,0) = \cos(x),\ v(x,0) = e^x \tag{106}$$

while the initial values for the filters were all set to zero. The kernel equations (60)–(65) were solved numerically in Matlab. The following input[2]

$$U(t) = \sin(t) + \sin(\sqrt{2}t) + \sin(\sqrt{3}t) \tag{107}$$

turned out to make the regressor $R(t)$ satisfy the PE requirements. The adaptation gain was set to

$$\Gamma = \text{diag}\{20, 20, 0.4, 0.4\} \tag{108}$$

[2]This is the same input as used for the simulations in [22].

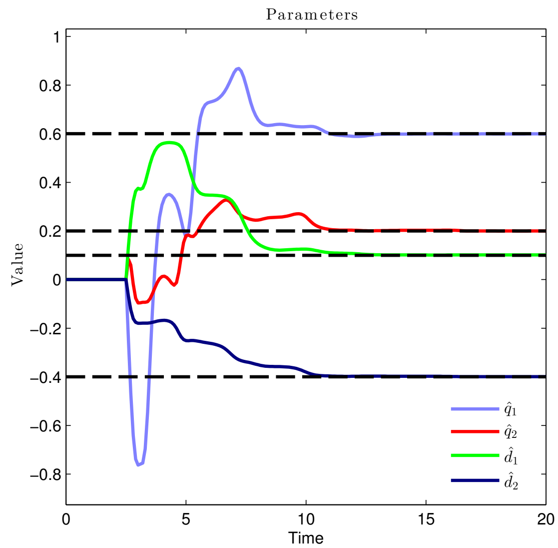

Fig. 1. Estimated and actual boundary parameters.

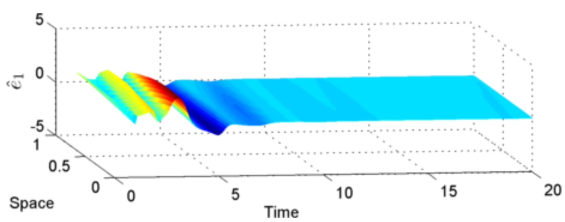

Fig. 2. Prediction error $\hat{e}_1(x,t)$.

The estimated parameters are found in Figure 1, with the "prediction error" $\hat{e}_1(x,t)$ generated from

$$\hat{e}(x,t) = u(x,t-\lambda^{-1}) - P(x,t)\hat{q} - W(x,t)\hat{d} \quad (109)$$

displayed in Figure 2 for $t \geq \lambda^{-1}$. Clearly, all the estimates converge to their true values within approximately 13 seconds, as well as the prediction error $\hat{e}_1(x,t)$ converging to zero. The prediction errors $\hat{e}_2(x,t)$ and $\hat{\epsilon}(x,t)$ look qualitatively the same as $\hat{e}_1(x,t)$.

VIII. Conclusion

We have developed an adaptive observer for estimating unknown parameters in the left boundary condition of a system of $n+1$ coupled, first-order 1-D hyperbolic PDEs, with measurements limited to the right boundary. The observer uses a set of filters that manage to express time-delayed system states as a linear combination of the filters and the unknown boundary parameters. A straight forward gradient or least squares adaptive law can then be used to estimate the unknown parameters. Proof of boundedness is given, and sufficient conditions for exponential parameter convergence are given. The theory is verified through a simulation.

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