Prediction-based control of linear systems subject to state-dependent state delay and multiple input-delays

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Abstract— **This paper presents a prediction-based controller strategy for linear systems subject to a state-dependent state delay and distinct constant input delays. We propose to compute corresponding predictions in cascade and use them in a nominal control law obtained from the input-delays free case. Using transport Partial Differential Equation (PDE) reformulations and backstepping transformations, we show that this control law compensates for the input delays in closed loop and provides nominal exponential stabilization. The mechanisms of this technique are illustrated on the dynamics of the mechanical vibrations in drilling, which has recently been described as a cutting process.**

I. INTRODUCTION

Prediction-based controllers (see [28], [1], [24]), also known as Smith predictors or reduction method, has been the focus of a large number of research directions in the last decade. This method, which was first introduced to compensate for a constant input delay affecting a linear time-invariant dynamics, has only recently been extended in various contexts: nonlinear plant [14], [19], [21], time-varying input delay [5], [25], [27], uncertain input delay [10] or dynamics [7], [23], systems involving both state and input delays [17], [2] or prediction implementation alternatives [26], [31]. However, all these methods consider a sole input delay.

Indeed, the problem of distinct delays significantly complicates the prediction design as it requires to compute different future state values on time horizons which could be larger than some input delays, making this computation non-causal at first sight. This is why the first design of a controller compensating for distinct input delays has only been proposed recently in the seminal paper [29]. In this work, the problem is formulated as a cascade between several first-order transport PDE (accounting for the distinct input delays) and an Ordinary Differential Equation (ODE) and solved using backstepping to design a prediction-based controller. The key idea of this prediction technique, extended in [6] to nonlinear systems and in [18] to systems involving both state and input delays, is to replace future inputs involved in the state predictions computation by their closed-loop expressions.

In this paper, we extend this technique to handle state-dependent state delays. This type of dependency has been considered in [3] for a specific class of state-dependent state delay systems which are not subject to input delay and in [4] which considers state-dependent input delays, resulting in a very intricate relation between the system state and the control inputs which is not involved here. These works have inspired the design we proposed in [8] to compensate for one input delay affecting a linear plant subject to a distributed state-dependent state-delay. Here, we follow the same methodology as in our previous work [8] and modify our prediction design in the spirit of [29] to account for multiple input-delays.

Following [8], our stability analysis is grounded on PDEs tools that were proposed lately to address input delay compensation (see [22], [20]) and were extended in [12] to handle the case of an additional distributed state-delay. Modeling both actuator and state delays as transport PDEs coupled with the original ODE, we rely on a backstepping transformation of the distributed input to analyze the closed-loop stability. To formulate the corresponding target system, one needs to study m implicit functional PDEs. While state delay is responsible for the implicit nature and distributed delay for the functional one, the consideration of m sets of PDEs follows from the fact that we are considering m distinct input delays. We then carry out an $\mathscr{L}_\infty$ analysis for the closed-loop system. This along with the prediction design is the main contribution of the paper.

The paper is organized as follows. In Section II, we introduce the problem under consideration before providing the prediction-based control we propose. Then, we present the stability analysis of the closed-loop dynamics in Section III. Finally, Section IV is devoted to an illustration of this technique mechanisms on the dynamics of the mechanical vibrations in drilling.

II. PROBLEM STATEMENT AND CONTROL DESIGN

We consider the problem of stabilizing the following (controllable) linear system subject to a distributed state-dependent state delay and constant distinct input delays

$$\dot{X}(t) = A_0 X(t) + A_1 \int_{t-D^S(X_t)}^{t} X(s)ds + \sum_{j=1}^{m} b_j U_j(t - D_j^I) \quad (1)$$

in which $X \in \mathbb{R}^n$, $U = [U_1 \dots U_m]^T \in \mathbb{R}^m$, the state-delay $D^S : \mathscr{C}([-\overline{D}, 0], \mathbb{R}^n) \to [0, \overline{D}]$ ($\overline{D} > 0$) is a continuously differentiable function, $D_1^I, \dots, D_m^I$ are constant input-delays such that $D_1^I < D_2^I < \dots < D_m^I$ and X_t denotes the function $X_t : s \in [-\overline{D}, 0] \mapsto X(t+s)$. Similarly, in the following, we denote $X_{t,S} : s \in [-D^S(X_t), 0] \mapsto X(t+s)$ and $U_{t,j} : s \in [-D_j^I, 0] \mapsto U(t+s)$ for $j = 1, \dots, m$. We assume[1] that $D_1^I \geq \overline{D}$ and X_0, U_0

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[1] This assumption is not restrictive. When the state delay can be larger than an input delay, the proposed control strategy remains unchanged, but predictions can be simplified.

are respectively continuously differentiable and continuous functions.

Before presenting our strategy to handle the distinct input delays, we make an assumption on the stabilization of the nominal input delays-free system.

Assumption 1: There exists a feedback law $\kappa : \mathscr{C}^0([-\bar{D},0],\mathbb{R}^n) \mapsto \mathbb{R}$ which is a class $\mathscr{C}^1$ function which can be upper-bounded by a linear mapping and such that the dynamics

$$\dot{X}(t) = A_0 X(t) + A_1 \int_{t-D^S(X_t)}^{t} X(s)ds + \sum_{j=1}^{m} b_j \kappa_j(X_{t,S}) \quad (2)$$

is globally exponentially stable, i.e., there exist a continuous functional $W : \mathscr{C}([-\bar{D},0],\mathbb{R}^n) \to \mathbb{R}$ and constants $C_1, C_2, C_3 > 0$ such that

$$C_1\|\varphi\|_\infty \leq W(\varphi) \leq C_2\|\varphi\|_\infty \quad (3)$$

$$|\partial_\varphi W(\varphi)| \leq C_3 \quad (4)$$

and, moreover, the functional W is differentiable along the trajectories of the closed-loop system (2) and

$$\dot{W}(t) \leq -W(t) \quad (5)$$

It is worth noting that requiring a linearly bounded feedback map is not demanding as we consider linear dynamics.

Even if this assumption can seem quite restrictive at first glance, it actually encompasses a large class of systems, including strict-feedforward systems as the following example illustrates it.

Example 1: Consider the plant

$$\dot{x}_1(t) = x_1(t) + \int_{t-D^S(X_t)}^{t} x_1(s)ds + x_2(t) \quad (6)$$

$$\dot{x}_2(t) = 2x_1(t) + x_2(t) + U_1(t) \quad (7)$$

$$\dot{x}_3(t) = U_2(t) \quad (8)$$

which is under the form (1) with $X = [x_1 \quad x_2 \quad x_3]^T$. The state delay D^S is a given known state-dependent function. The subsystems (6)–(7) and (8) are independent. We first consider (8) and choose

$$\kappa_2(X_t) = -x_3(t) \quad (9)$$

Now, we focus on the subsystem (x_1, x_2). Taking x_2 as a virtual input in (6) to map it into the target dynamics $\dot{x}_1(t) = -x_1(t)$ leads to the choice

$$\begin{aligned}\kappa_1(X_t) &= -[x_2(t) - v(t)] - 2x_1(t) - x_2(t) + \dot{v}(t) \quad (10)\\ &= -7x_1(t) - 2x_2(t) + x_1(t - D^S(X_t)) - 3\int_{t-D^S(X_t)}^{t} x_1(s)ds\end{aligned}$$

in which

$$v(t) = -2x_1(t) - \int_{t-D^S(X_t)}^{t} x_1(s)ds \quad (11)$$

The control law $\kappa(X_t) = [\kappa_1(X_t) \quad \kappa_2(X_t)]^T$ defined through (9)–(10) satisfies Assumption 1 as it can be upper-bounded by a linear mapping, it is of class $\mathscr{C}_\infty$ and the closed-loop system corresponding to $U(t) = \kappa(X_t)$ is

$$\dot{x}_1(t) = -x_1(t) + x_2(t) - v(t) \quad (12)$$

$$\dot{x}_2(t) - \dot{v}(t) = -(x_2(t) - v(t)) \quad (13)$$

$$\dot{x}_3(t) = -x_3(t) \quad (14)$$

which is exponentially stable.

We are now ready to detail the control design of (1). We define

$$\tilde{D}^I_{ij} = D^I_i - D^I_j, \quad i,j = 1,\dots,m \quad (15)$$

and consider the distributed state predictions

$$P^1(t,\tau) = \quad (16)$$
$$\begin{cases} X(\tau + D^I_1) & \text{if } t - \overline{D} - D^I_1 \leq \tau \leq t - D^I_1 \\ e^{A_0(\tau + D^I_1 - t)}X(t) + \displaystyle\int_{t-D^I_1}^{\tau} e^{A_0(\tau - s)}\left[A_1\int_{s-D^S(P^1_s)}^{s} P^1(t,\xi)d\xi + \sum_{j=1}^{m} b_j U_j(s - \tilde{D}^I_{j1})\right]ds & \text{if } t - D^I_1 \leq \tau \leq t \end{cases}$$

$$P^2(t,\tau) = \quad (17)$$
$$\begin{cases} P^1(t,\tau + \tilde{D}^I_{21}) & \text{if } t - \overline{D} - \tilde{D}^I_{21} \leq \tau \leq t - \tilde{D}^I_{21} \\ e^{A_0(\tau + \tilde{D}^I_{21} - t)}P^1(t,t) + \displaystyle\int_{t-\tilde{D}^I_{21}}^{\tau} e^{A_0(\tau - s)}\left[A_1\int_{s-D^S(P^2_s)}^{s} P^2(t,\xi)d\xi + b_1\kappa_1(P^2_{s,S}) + \sum_{j=2}^{m} b_j U_j(s - \tilde{D}^I_{j2})\right]ds & \text{if } t - \tilde{D}^I_{21} \leq \tau \leq t \end{cases}$$

$$P^3(t,\tau) = \quad (18)$$
$$\begin{cases} P^2(t,\tau + \tilde{D}^I_{32}) & \text{if } t - \overline{D} - \tilde{D}^I_{32} \leq \tau \leq t - \tilde{D}^I_{32} \\ e^{A_0(\tau + \tilde{D}^I_{31} - t)}P^2(t,t) + \displaystyle\int_{t-\tilde{D}^I_{32}}^{\tau} e^{A_0(\tau - s)}\left[A_1\int_{s-D^S(P^3_s)}^{s} P^3(t,\xi)d\xi + \sum_{j=1}^{2} b_j\kappa_j(P^3_{s,S}) + \sum_{j=3}^{m} b_j U_j(s - \tilde{D}^I_{j3})\right]ds & \text{if } t - \tilde{D}^I_{32} \leq \tau \leq t \end{cases}$$

$$\vdots$$

$$P^m(t,\tau) = \quad (19)$$
$$\begin{cases} P^{m-1}(t,\tau + \tilde{D}^I_{m(m-1)}) & \text{if } t - \overline{D} - \tilde{D}^I_{m(m-1)} \leq \tau \leq t - \tilde{D}^I_{m(m-1)} \\ e^{A_0(\tau + \tilde{D}^I_{m(m-1)} - t)}P^{m-1}(t,t) + \displaystyle\int_{t-\tilde{D}^I_{m(m-1)}}^{\tau} e^{A_0(\tau - s)} \times \left[A_1\int_{s-D^S(P^m_s)}^{s} P^m(t,\xi)d\xi + \sum_{j=1}^{m-1} b_j\kappa_j(P^m_{s,S}) + b_m U_m(s)\right]ds & \text{if } t - \tilde{D}^I_{m(m-1)} \leq \tau \leq t \end{cases}$$

We now use these predictions as argument for the nominal input-delay free control law in lieu of the original state

$$U_j(t) = \kappa_j(P^j_{t,S}), \quad j = 1,\dots,m \quad (20)$$

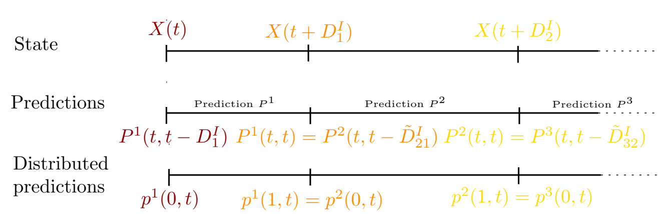

Fig. 1. Schematic view of the predictions defined in (16)–(17) and corresponding future values of the state. Distributed predictions involved in the proof of Theorem 1 and defined in (36),(39) are also depicted.

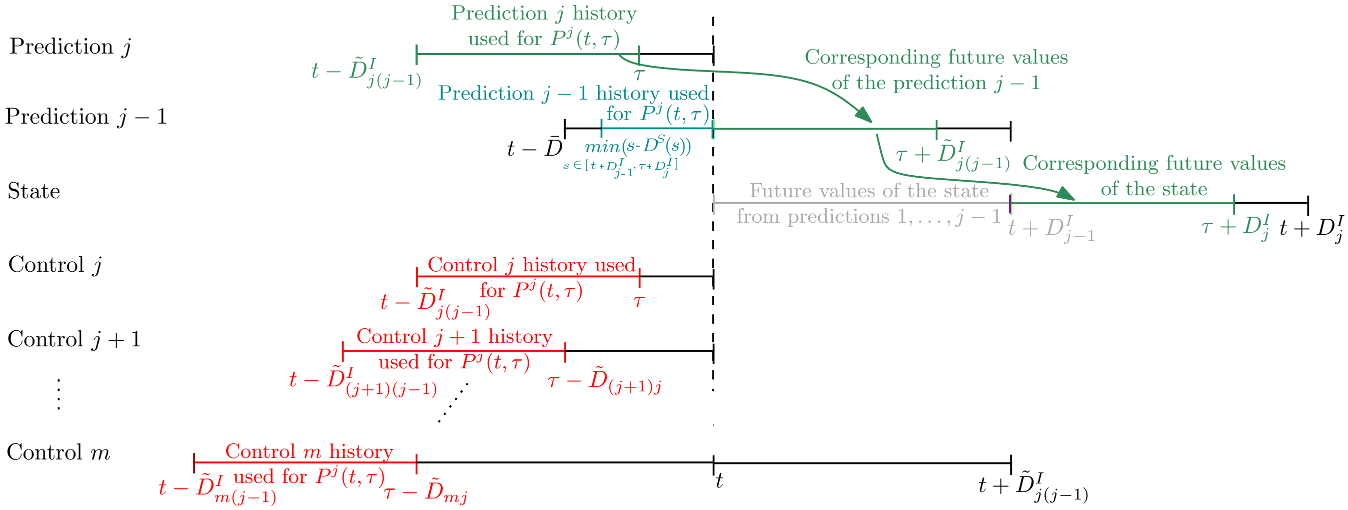

Fig. 2. Schematic view of the variables involved in the computation of the prediction $P^j(t,\tau)$. Consequently, the computation of $P^m(t,\tau)$ involves the history of X over $[\min_{s\in[t,\tau+D^I_m]}(s-D^S(s)),t]$ (which is included in $[t-\overline{D},t]$), of U_j over $[t-D^I_j,t]$ for $j=1,\ldots,m-1$ and of U_m over $[t-D^I_m,\tau]$.

and have the corresponding result.

Theorem 1: Consider the closed-loop system consisting of (1) satisfying Assumption 1 and the control law (20) involving the predictions (16)–(19). Define the functional

$$\Gamma(t)=\max_{s\in[-\overline{D},0]}|X(t+s)|+\sum_{j=1}^{m}\max_{s\in[-D^I_j,0]}|U_j(t+s)| \tag{21}$$

There exist $R,\rho>0$ such that, for $(X_0,(U_1)_{0,1},\ldots,(U_m)_{0,m})\in\mathscr{C}^0([-\overline{D},0],\mathbb{R}^n)\times\mathscr{C}^0([-D^I_1,0],\mathbb{R})\times\ldots\times\mathscr{C}^0([-D^I_m,0],\mathbb{R})$,

$$\Gamma(t)\leq R\Gamma(0)e^{-\rho t},\quad t\geq 0 \tag{22}$$

In order to understand properly the choice of this control law, we provide several comments next, before detailing the proof of this result.

First, it is crucial to remark that the function[2] $P^1(t,\cdot)$ defined in (16) is a D^I_1 units of time ahead prediction of X_t, the state history over a time horizon $\overline{D}$. This follows from the integration of (1) between t and $\tau+D^I_1$ which is

$$X(\tau+D^I_1)=e^{A_0(\tau+D^I_1-t)}X(t)+\int_t^{\tau+D^I_1}e^{A_0(\tau+D^I_1-t)} \times\left(A_1\int_{s-D^s(X_s)}^{s}X(\xi)d\xi+\sum_{j=1}^{m}b_jU_j(s-D^I_j)\right) \tag{23}$$

Performing a change of variable under the main integral, one indeed obtains (16) and formally infers that $P^1(t,\tau)=X(\tau+D^I_1)$. Similarly, the function $P^2(t,\cdot)$ defined in (17) is a D^I_2 units of time ahead prediction of X_t, obtained by integrating (1) between $t+D^I_1$ and $\tau+D^I_1+\tilde{D}^I_{21}=\tau+D^I_2$. It is worth noticing that the prediction P^2 involves future values of U_1 which are thus replaced by their closed-loop expressions, which have already been fixed via (20). Thus, iteratively, the functions $P^j(t,\cdot)$ defined through (16)–(19) are D^I_j units of time ahead predictions of X_t. These relations are summarized in Fig. 1. From there, the choice of the control law naturally follows as plugging (20) into the original dynamics (1), one informally obtains (2) which is assumed to be stable. In other words, the role of these cascaded predictions is to cancel each of the input delays iteratively to recover an input-delays free dynamics. In short, multiple input-delays are compensated for by:

- m different predictions, accounting for the m distinct input delays;
- predictions that are defined sequentially (prediction j uses prediction $j-1$ as a starting point, as illustrated in Fig 2) and replacing future input values by their closed-loop expressions (20).

Second, it is worth mentioning that, the implicit control law (20) is actually well-defined as solution of an integral formulation of (1) (Volterra integral equation). Furthermore, it can also be computed, relying on suitable discretization schemes (see [15], [16], [30] for a sole input-delay case) and this computation is causal as it only requires the knowledge of the state X over the time horizon $[t-\overline{D},t]$ and the inputs U_j over the respective time horizons $[t-D^I_j,t]$ $(j=1,\ldots,m)$. This fact is depicted in Fig. 2.

Finally, it is worth understanding that, contrary to [4], we do not need to impose any restriction on the state-dependent delay rate here and thus to limit our result to a local one. Indeed, we consider that this state-dependency only affects the state delay. Consequently, the predictions (16)–(19) are not impacted by the state-delay rate, which can be arbitrarily large a priori and in particular can vary faster than the absolute time.

We now provide the proof of this theorem.

III. Stability Analysis – Proof of Theorem1

In the sequel, for the sake of conciseness, we sometimes write $D^S(t)=D^S(X_t)$ and $\dot{D}^S(t)=\frac{dD^S}{dX_t}\cdot\dot{X}_t$.

[2]We write P^j as functions of two arguments to emphasize the fact that predictions should be computed by incorporating measured delayed states in the definition (16)–(19) as opposed to the integration method in [2]. Formally, the two formulations are equivalent but the formulation we propose here should improve the robustness of the controller to model mismatch.

We provide a proof based on PDE considerations in order to provide tools that could then be used in future works for robustness analysis.

A. PDEs reformulations

To analyze the stability of the closed-loop system, we propose to reformulate delays as transport PDEs and define the distributed variables

$$\zeta(x,t) = X(t+D^S(t)(x-1)) \tag{24}$$

$$\eta(x,t) = X(t-\bar{D}+(\bar{D}-D^S(t))x) \tag{25}$$

$$\bar{\zeta}(x,t) = \begin{cases} \zeta\left((x-1)\dfrac{\bar{D}}{D^S(t)}+1,t\right) \text{if } x \geq 1-\dfrac{D^S(t)}{\bar{D}} \\ \eta\left(\dfrac{\bar{D}}{\bar{D}-D^S(t)}x,t\right) \text{if } x \leq 1-\dfrac{D^S(t)}{\bar{D}} \end{cases} \tag{26}$$

$$u_j(x,t) = U_j(t+D_j^I(x-1)), \quad j=1,\dots,m \tag{27}$$

in which, in details, ζ accounts for the history of the state X over a time window of varying length $D^S(X_t)$, η for the complementary history of X over $[t-\bar{D},t-D^S(X_t)]$, $\bar{\zeta}$ for the entire history over $[t-\overline{D},t]$ and u_j for the history of the input j, U_j, over $[t-D_j^I,t]$.

The plant (1) can then be reformulated as the following PDE-ODE cascade

$$\dot{X}(t) = A_0X(t)+A_1D^S(t)\int_0^1 \zeta(x,t)dx+\sum_{j=1}^m b_ju_j(0,t) \tag{28}$$

$$D^S(t)\partial_t\zeta(x,t) = (1+\dot{D}^S(t)(x-1))\partial_x\zeta(x,t) \tag{29}$$

$$\zeta(1,t) = X(t) \tag{30}$$

$$(\bar{D}-D^S(t))\partial_t\eta(x,t) = (1-\dot{D}^S(t)x)\partial_x\eta(x,t) \tag{31}$$

$$\eta(1,t) = \zeta(0,t) \tag{32}$$

$$D_j^I\partial_tu_j(x,t) = \partial_xu_j(x,t), \quad j=1,\dots,m \tag{33}$$

$$u_j(1,t) = U_j(t) \tag{34}$$

and, in addition, it follows that

$$\bar{D}\partial_t\bar{\zeta}(x,t) = \partial_x\bar{\zeta}(x,t) \quad \text{with} \quad \bar{\zeta}(1,t)=X(t) \tag{35}$$

Now, define distributed predictions

$$p^1(x,t) = e^{A_0D_1^Ix}X(t)+D_1^I\int_0^x e^{A_0D_1^I(x-y)}\Big[A_1D_0^S(\bar{\chi}^1(y,\cdot,t)) \times\int_0^1\chi^1(y,\xi,t)d\xi+\sum_{j=1}^m b_ju_j\left(\frac{D_1^I}{D_j^I}y,t\right)\Big]dy \tag{36}$$

$$\bar{\chi}^1(x,y,t) = \tag{37}$$
$$\begin{cases} \bar{\zeta}\left(y+\dfrac{D_1^I}{\overline{D}}x,t\right) & \text{if} \quad xD_1^I+\overline{D}(y-1)\leq 0 \\ p^1\left(x+\dfrac{\overline{D}}{D_1^I}(y-1),t\right) & \text{if} \quad xD_1^I+\overline{D}(y-1)\geq 0 \end{cases}$$

$$\chi^1(x,y,t) = \tag{38}$$
$$\begin{cases} X(t+D_1^Ix+D_0^S(\bar{\chi}_1(x,\cdot,t))(y-1)) \\ \qquad \text{if} \quad xD_1^I+D_0^S(\bar{\chi}_1(x,\cdot,t))(y-1)\leq 0 \\ p^1\left(x+\dfrac{D_0^S(\bar{\chi}_1(x,\cdot,t))}{D_1^I}(y-1),t\right) \\ \qquad \text{if} \quad xD_1^I+D_0^S(\bar{\chi}_1(x,\cdot,t))(y-1)\geq 0 \end{cases}$$

and, for $j=2,\dots,m$,

$$p^j(x,t) = e^{A_0\tilde{D}_{j(j-1)}^Ix}p^{j-1}(1,t)+\tilde{D}_{j(j-1)}^I\int_0^x e^{A_0\tilde{D}_{j(j-1)}^I(x-y)} \times\Big[A_1D_0^S(\bar{\chi}^j(y,\cdot,t))\int_0^1\chi^j(y,\xi,t)d\xi \tag{39}$$
$$+\sum_{i=1}^{j-1}b_i\kappa_i^0(\chi^j(y,\cdot,t))+\sum_{i=j}^m b_iu_i\left(\frac{\tilde{D}_{j(j-1)}^I}{D_i^I}y+\frac{D_{j-1}^I}{D_i^I},t\right)\Big]dy$$

$$\bar{\chi}^j(x,y,t) = \tag{40}$$
$$\begin{cases} \bar{\chi}^{j-1}\left(1,y+\dfrac{\tilde{D}_{j(j-1)}^I}{\overline{D}}x,t\right) & \text{if} \quad x\tilde{D}_{j(j-1)}^I+\overline{D}(y-1)\leq 0 \\ p^j\left(x+\dfrac{\overline{D}}{\tilde{D}_{j(j-1)}^I}(y-1),t\right) & \text{if} \quad x\tilde{D}_{j(j-1)}^I+\overline{D}(y-1)\geq 0 \end{cases}$$

$$\chi^j(x,y,t) = \tag{41}$$
$$\begin{cases} p^{j-1}(1,t+\tilde{D}_{j(j-1)}^Ix+D_0^S(\bar{\chi}^j(x,\cdot,t))(y-1)) \\ \qquad \text{if} \quad x\tilde{D}_{j(j-1)}^I+D_0^S(\bar{\chi}^j(x,\cdot,t))(y-1)\leq 0 \\ p^j\left(x+\dfrac{D_0^S(\bar{\chi}^j(x,\cdot,t))}{\tilde{D}_{j(j-1)}^I}(y-1),t\right) \\ \qquad \text{if} \quad x\tilde{D}_{j(j-1)}^I+D_0^S(\bar{\chi}^j(x,\cdot,t))(y-1)\geq 0 \end{cases}$$

in which we introduced the function D_0^S defined as $D_0^S(\bar{\chi}^1(0,\cdot,t)) = D^S(X_{t,S})$.

The distributed predictions p^j (36),(39) ($j=1,\dots,m$) are simply distributed reformulations of the predictions (16)–(19). This relation is illustrated in Fig. 1. In details, $p^j(x,t)$ accounts for $X(t+D_{j-1}^I+x\tilde{D}_{j(j-1)}^I)$ or, equivalently, $P^j(t,t+\tilde{D}_{j(j1)}^I(x-1))$ while $\bar{\chi}^j(x,\cdot,t)$ and $\chi^j(x,\cdot,t)$ represent the history of $p^j(x,t)$ over $[t-\bar{D},t]$ and $[t-D^S(t+D_{j-1}^I+x\tilde{D}_{j(j-1)}^I),t]$ respectively. The role of these last two variables is to describe the state-dependent state delay. This characteristic feature of the dynamics at stake complicates the analysis as p^j,χ^j and $\bar{\chi}^j$ are defined depending on each other and should thus be studied together.

Finally, define the following backstepping transformation

$$w_j(x,t) = u_j(x,t)-\kappa_j^0\left(\chi^{i+1}\left(\frac{D_j^I}{\tilde{D}_{(i+1)i}^I}\left(x-\frac{D_i^I}{D_j^I}\right),\cdot,t\right)\right)$$
$$\frac{D_i^I}{D_j^I}\leq x\leq\frac{D_{i+1}^I}{D_j^I},\ i=0,\dots,j-1 \tag{42}$$

in which we introduced the notations $\kappa^0(\zeta(\cdot,t)) = \kappa(X_{t,S})$ and $D_0^I=0$. We then have the following result, proved in Appendix.

Lemma 1: The infinite-dimensional backstepping transformation (42) together with the control law (20) transform (28)–(34) into the target system

$$\dot{X}(t) = A_0X(t)+A_1D^S(t)\int_0^1\zeta(x,t)dx +\sum_{j=1}^m b_j[\kappa_j^0(\zeta(\cdot,t))+w^j(0,t)] \tag{43}$$

$$D^S(t)\partial_t\zeta(x,t) = (1+\dot{D}^S(t)(x-1))\partial_x\zeta(x,t) \quad (44)$$

$$\zeta(1,t) = X(t) \quad (45)$$

$$(\bar{D}-D^S(t))\partial_t\eta(x,t) = (1-\dot{D}^S(t)x)\partial_x\eta(x,t) \quad (46)$$

$$\eta(1,t) = \zeta(0,t) \quad (47)$$

$$D_j^I\partial_t w_j(x,t) = \partial_x w_j(x,t), \quad j=1,\dots,m \quad (48)$$

$$w_j(1,t) = 0 \quad (49)$$

The goal of this backstepping variable is to simplify the Lyapunov analysis by providing suitable boundary conditions equal to zero in (49). Note that these boundary conditions follow from the fact that the choice of (42) is consistent with the choice of the control law (20). In details, in (42), κ_j is evaluated with the distributed prediction i ($1 \le i \le j$) depending on the location of the variable x between 0 and 1: the larger x is, the older the prediction needs to be and thus the bigger the number of the considered prediction.

Note that we made the choice to introduce all distributed variables in this section as normalized, i.e. with $x \in [0,1]$. This complicates the expressions of these variable but introduces a linear delay parameterization in (29),(31),(33),(48) which should be useful, e.g., to carry out a delay-robustness analysis as performed in [11], [12].

B. Lyapunov analysis

Consider the following Lyapunov functional candidate

$$V_p(t) = \left(\mu_0 + 1 - \frac{1}{2p}\right)V_0(t)^{2p} + b^{2p}\sum_{j=1}^m D_j^I \int_0^1 e^{2p\mu_0 x} w_j(x,t)^{2p}dx \quad (50)$$

in which V_0 has been introduced in Assumption 1, $p \in \mathbb{N}^*$ and $\mu_0, b > 0$. Taking a time-derivative, one gets

$$\begin{aligned}\dot{V}_p(t) = &-(2p(\mu_0+1)-1)V_0(t)^{2p} \quad (51)\\ &+(2p(\mu_0+1)-1)V_0(t)^{2p-1}\partial_\varphi V_0(X_t)\sum_{j=1}^m b_j w_j(0,t)\\ &-b^{2p}\sum_{j=1}^m w_j(0,t)^{2p} - b^{2p}2p\mu_0\sum_{j=1}^m\int_0^1 e^{2p\mu_0 x}w_j^{2p}(x,t)dx\end{aligned}$$

Using Young inequality and (4), one obtains

$$\begin{aligned}\dot{V}_p(t) = &-2p\mu_0 V_0(t)^{2p} - b^{2p}2p\mu_0\sum_{j=1}^m\int_0^1 e^{2p\mu_0 x}w_j^{2p}(x,t)dx\\ &-\sum_{j=1}^m\left(b^{2p}-(\mu_0+1)^{2p}|C_3 b_j|^{2p}\right)w_j(0,t)^{2p} \quad (52)\end{aligned}$$

Consequently, choosing

$$b > \max_{j=1,\dots,m}(\mu_0+1)|C_3 b_j| \quad (53)$$

it follows that

$$\dot{V}_p(t) \le -2p\eta V_p(t) \quad (54)$$

in which $\eta = \min\left\{\frac{\mu_0}{\mu_0+1}, \frac{1}{D_m^I}\right\}$ and thus

$$\begin{aligned}V_p(t)^{\frac{1}{2p}} \le e^{-\eta t}\Bigg(&\left(\mu_0+1-\frac{1}{2p}\right)^{\frac{1}{2p}}V_0(0) \quad (55)\\ &+b\sum_{j=1}^m\left(D_j^I\int_0^1 e^{2p\mu_0 x}w_j(x,0)^{2p}dx\right)^{\frac{1}{2p}}\Bigg)\end{aligned}$$

This gives

$$\begin{aligned}&\left(\mu_0+1-\frac{1}{2p}\right)^{\frac{1}{2p}}V_0(t) + b\sum_{j=1}^m\left(D_j^I\int_0^1 e^{2p\mu_0 x}w_j(x,0)^{2p}dx\right)^{\frac{1}{2p}}\\ &\le 2^m e^{-\eta t}\Bigg(\left(\mu_0+1-\frac{1}{2p}\right)^{\frac{1}{2p}}V_0(0)\\ &\qquad + b\sum_{j=1}^m\left(D_j^I\int_0^1 e^{2p\mu_0 x}w_j(x,0)^{2p}dx\right)^{\frac{1}{2p}}\Bigg) \quad (56)\end{aligned}$$

Taking the limit as p tends to infinity, one obtains

$$\begin{aligned}&V_0(t) + b\sum_{j=1}^m \max_{x\in[0,1]} e^{\mu_0 x}|w_j(x,t)| \quad (57)\\ &\le e^{-\eta t}\left(V_0(0) + b\sum_{j=1}^m \max_{x\in[0,1]} e^{\mu_0 x}|w_i(x,0)|\right)\end{aligned}$$

Finally, using (3), the fact that κ can be upper-bounded by a linear mapping from Assumption 1 and applying Young and Cauchy-Schwarz inequalities to the backstepping transformation (42), one obtains the desired result.

IV. Illustrative Example – Suppression of Mechanical Vibrations in Drilling

In this section, we illustrate the proposed control strategy and its merit on the dynamics of mechanical vibrations in drilling.

Mechanical vibrations are an important source of Non-Productive Time (NPT) and failure in the oil drilling industry. The operator imposes a force and rotating velocity at the surface, as depicted on Fig. 3. These are transmitted to the Bottom Hole Assembly (BHA) several kilometers downhole. This device, holding the drillbit, creates the borehole. It can be considered as a lumped oscillating mass, subject to axial and torsional displacement waves which are traveling up and down the drillstring at a finite velocity.

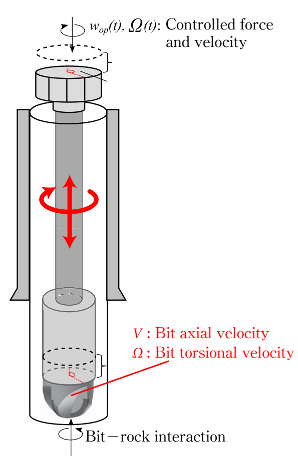

Fig. 3. Schematic view of drilling facilities.

The interaction of the drillbit with the rock can be described by a cutting process [13]: both the torque and weight-on-bit are proportional to the *depth of cut*, defined as the vertical displacement of the bit over one revolution. This gives rise to a state-dependent state delay model with distinct input delay. Indeed, the following equations (see [8], [13]) describe the deviation of the BHA state from an equilibrium

$$\dot{V}(t) = -\alpha V(t) - \beta \int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} V(s)ds - \gamma \int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} \Omega(s)ds + \alpha U_1(t - L_p/c_a) \quad (58)$$

$$\dot{\Omega}(t) = -\alpha'\Omega(t) - \beta \int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} V(s)ds - \gamma \int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} \Omega(s)ds + \alpha' U_2(t - L_p/c_a) \quad (59)$$

in which $U_1(t) = V_R(t) + \tilde{w}_{op}(t)$ and $U_2(t) = \Omega_R(t) + \tilde{\Omega}(t)$ and the state-delay $\tilde{t}_N$ satisfies the following implicit equation

$$\int_{t-\bar{t}_N-\tilde{t}_N(t)}^{t} \Omega(s)ds + \Omega_0 \tilde{t}_N(t) = 0 \quad (60)$$

All states and parameters are defined in Table I, with their dependence on time t and space x. We consider that the top and bottom velocities[3] are measured.

Using the fact that the axial wave velocity is larger than the torsional one ($c_a > c_\tau$), one can observe that equations (58)–(60) correspond to the plant considered in this paper in (1) with [4] $D^S(X_t) = \bar{t}_N + \tilde{t}_N(t)$, $D_1^I = L_P/c_a$ and $D_2^I = L_P/c_\tau > D_1^I$. Finally, Assumption 1 is satisfied here with the feedback law

$$\kappa(X_t) = -A_1 \int_{t-D^S(X_t)}^{t} X(s)ds - K_0 X(t) \quad (61)$$

in which K_0 is a given matrix such that the closed-loop matrix dynamics $A_0 + BK_0$ is Hurwitz.

Figures 4 and 5 picture simulations where the proposed controller (20) is used[5] to stabilize the equilibrium corresponding to a nominal rotational velocity $\Omega_0 = 120$ rev/min. We pick K_0 as a zero matrix as the parameters α and α' are positive and thus A_0 is already Hurwitz. We compare this controller with the one proposed in [8] which voluntarily delayed the control U_1 to obtain a unique input delay for the two control paths. Both controller are turned on after 10s. One can observe that, after an oscillatory behavior in open-loop (before 10s), both controllers achieve exponential stabilization towards the desired equilibrium. The main difference between both controllers can be observed in Fig. 4. When the first prediction P^1 is used in the control law acting on the axial velocity, this one reacts faster than with one sole prediction (that is, $D_2^I - D_1^I$ units of time before). Therefore, transient performances are improved. In turn, transient performances are slightly improved for the torsional velocity before the second control action, using P^2, kicks in. However, this action is modest as U_2 is still affected by D_2^I in both cases.

[3]While the top velocity is conventionally measured, the bottom one may actually not be available in practice. Future works will investigate the interest of extending the techniques employed in [9] in this case.

[4]Note that the state delay is not upper-bounded here, a priori. Future works should investigate this point.

[5]Note that here the implementation of both control laws requires to suppress the reflections at the top (through V_R and Ω_R) for the two displacement waves. This cancellation is unlikely to be exact in practice.

V. CONCLUSION

In this work, we proposed an extension of multiple input delays compensation techniques for linear systems to handle additional state-dependent state delays. Future works will focus on the robustness properties to model uncertainties of this class of techniques, which largely ground on the exact knowledge of the systems dynamics.

Symbol	Unit	Description
$V(t)$	m.s^{-1}	Bit axial velocity
$\Omega(t)$	rad.s^{-1}	Bit torsional velocity
α, α', β, β', γ, γ'		Bit-rock interaction law parameters (positive)
$\bar{t}_N$	s	Nominal state-delay at the bit
$\tilde{t}_N$	s	Deviation of the state-delay at the bit
Ω_0	rad.s^{-1}	Nominal rotational velocity
$\tilde{w}_{op}(t)$	m.s^{-1}	(Scaled) weight applied by the operator
$\tilde{\Omega}(t)$	rad.s^{-1}	Rotational velocity applied by the operator
$V_R(t)$	m.s^{-1}	Top axial velocity
$\Omega_R(t)$	rad.s^{-1}	Top rotational velocity
L_p	m	Drillstring length
c_a	m.s^{-1}	Axial wave velocity
c_τ	m.s^{-1}	Torsional wave velocity

TABLE I

STATES AND PARAMETERS OF THE MECHANICAL VIBRATIONS MODEL (58)–(60)

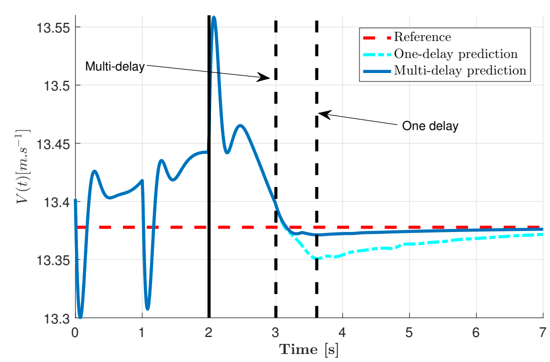

Fig. 4. Axial velocity of the BHA. The solid black line corresponds to the time instant when the controller is switched on. The dashed black line corresponds to the time instant when the control kicks in.

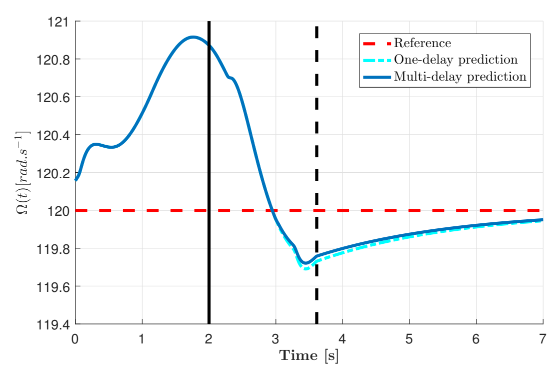

Fig. 5. Torsional velocity of the BHA. The solid black line corresponds to the time instant when the controller is switched on. The dashed black line corresponds to the time instant when the control kicks in.

APPENDIX A – PROOF OF LEMMA 1

We start by noticing that, for $j=1$, $p^j(x,t)=P^j(t,t+\tilde{D}^I_{j(j-1)}(x-1))$, that $\bar{\chi}^j(x,y,t)=P^j(t,t+\tilde{D}^I_{j(j-1)}(x-1)+\bar{D}(y-1))$ and that $\chi^j(x,y,t)=P^j(t,t+\tilde{D}^I_{j(j-1)}(x-1)+D^S_0(\bar{\chi}(x,\cdot,t))(y-1))$ for $(x,y)\in|0,1]^2$ and thus that the same properties hold for all $j=1,\dots,m$. From (42) evaluated for $x=1$ and $x=0$, one thus directly gets that $w_j(1,t)=u_j(1,t)-\kappa^0_j(\chi^j(1,\cdot,t))=0$ and that $w_j(0,t)=u_j(0,t)-\kappa^0_j(\zeta(\cdot,t))$.

Following the cascade structure of the predictions definition, we proceed by iterations to prove that, for $j=1,\dots,m$,

$$\tilde{D}^I_{j(j-1)}\partial_t\chi^j(x,y,t)-\partial_x\chi^j(x,y,t)=0 \tag{62}$$

$$\tilde{D}^I_{j(j-1)}\partial_t\bar{\chi}^j(x,y,t)-\partial_x\bar{\chi}^j(x,y,t)=0 \tag{63}$$

$$\begin{aligned}\partial_t p^j(1,t)&=A_0p^j(1,t)+A_1D^0_S(\bar{\chi}^j(1,\cdot,t))\int_0^1\chi^j(1,\xi,t)d\xi\\&+\sum_{i=1}^{j-1}b_i\kappa_i(\chi^j(1,\cdot,t))+\sum_{i=j}^{m}b_iu_i\left(\frac{D^I_j}{D^I_i},t\right)\end{aligned} \tag{64}$$

We start by studying the dynamics of p^1, following the same steps as in the proof of [8]. Taking space- and time-derivatives of (36), one gets

$$\begin{aligned}&\partial_t p^1(x,t)=\\&e^{A_0D^I_1x}[A_0X(t)+A_1D^S(t)\int_0^1\zeta(x,t)dx+\sum_{j=1}^m b_ju_j(0,t)]\\&+D^I_1\int_0^x e^{A_0D^I_1(x-y)}\left[A_1\frac{dD^S_0}{d\bar{\chi}^1}\cdot\partial_t\bar{\chi}^1(y,\cdot,t)\int_0^1\chi^1(y,\xi,t)d\xi\right.\\&\left.+A_1D^S_0(\bar{\chi}^1(y,\cdot,t)\int_0^1\partial_t\chi^1(y,\xi,t)+\sum_{j=1}^m b_j\partial_tu_j\left(\frac{D^I_1}{D^I_j}y,t\right)\right]dy\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\partial_x p^1(x,t)=\\&A_0D^I_1e^{A_0D^I_1x}X(t)+D^I_1A_1D^S_0(\bar{\chi}^1(x,\cdot,t))\int_0^1\chi^1(x,\xi,t)d\xi\\&+D^I_1\sum_{j=1}^m b_ju_j\left(\frac{D^I_1}{D^I_j}x,t\right)+D^I_1\int_0^x A_0D^I_1e^{A_0D^I_1(x-y)}\times\\&\left[A_1D^S_0(\bar{\chi}^1(y,\cdot,t))\int_0^1\chi^1(y,\xi,t)d\xi+\sum_{j=1}^m b_ju_j\left(\frac{D^I_1}{D^I_j}y,t\right)\right]dy\\&=D^I_1e^{A_0D^I_1x}\left[A_0X(t)+A_1D^S_0(\bar{\chi}^1(0,\cdot,t))\int_0^1\chi^1(0,\xi,t)d\xi\right.\\&\left.+\sum_{j=1}^m b_ju_j(0,t)\right]+D^I_1\int_0^x e^{A_0D^I_1(x-y)}\left[A_1\frac{dD^S_0}{d\bar{\chi}^1}\cdot\partial_x\bar{\chi}^1(y,\cdot,t)\right.\\&\times\int_0^1\chi^1(y,\xi,t)d\xi+A_1D^S_0(\bar{\chi}^1(y,\cdot,t))\int_0^1\partial_x\chi^1(y,\xi,t)d\xi\\&\left.+\sum_{j=1}^m b_j\frac{D^I_1}{D^I_j}\partial_xu_j\left(\frac{D^I_1}{D^I_j}y,t\right)\right]dy\end{aligned} \tag{66, 67}$$

in which we used an integration by parts. Observing that $\chi^1(0,\cdot,t)=\zeta(\cdot,t)$, that $\bar{\chi}^1(0,\cdot,t)=\bar{\zeta}(\cdot,t)$ and thus that $D^S_0(\bar{\chi}^1(0,\cdot,t))=D^S(t)$ and using (33), one obtains

$$\begin{aligned}&D^I_1\partial_tp^1(x,t)-\partial_xp^1(x,t)=D^I_1\int_0^x e^{A_0D^I_1(x-y)}\times\\&\left[A_1\frac{dD^S_0}{d\bar{\chi}}\cdot(D^I_1\partial_t\bar{\chi}^1(y,\cdot,t)-\partial_x\bar{\chi}^1(y,\cdot,t))\int_0^1\chi^1(y,\xi,t)d\xi\right.\\&\left.+A_1D^S_0(\bar{\chi}^1(y,\cdot,t))\int_0^1(D^I_1\partial_t\chi^1(y,\xi,t)-\partial_x\chi^1(y,\xi,t))d\xi\right]dy\end{aligned} \tag{68}$$

From there, using the exact same arguments as in [8], one obtains that (62)–(63) hold for $j=1$. Also, from (62) evaluated for $y=1$, one concludes that $D^I_1\partial_tp^1(1,t)=\partial_xp^1(1,t)$. Using this fact along with (66) evaluated at $x=1$, one concludes that (64) for $j=1$ also holds.

Next, we assume that (62)–(64) hold for $i=1,\dots,j-1$. Taking time- and space-derivatives of (39), one gets

$$\begin{aligned}&\partial_tp^j(x,t)=e^{A_0\tilde{D}^I_{j(j-1)}x}\partial_tp^{j-1}(1,t)+\tilde{D}^I_{j(j-1)}\int_0^x e^{A_0\tilde{D}^I_{j(j-1)}(x-y)}\\&\times\left[A_1\frac{dD^S_0}{d\bar{\chi}}\cdot\partial_t\bar{\chi}^j(y,\cdot,t)\int_0^1\chi^j(y,\xi,t)d\xi\right.\\&+A_1D^S_0(\bar{\chi}^j(y,\cdot,t))\int_0^1\partial_t\chi^j(y,\xi,t)d\xi\\&\left.+\sum_{i=1}^{j-1}b_i\frac{d\kappa^0_i}{d\chi}\cdot\partial_t\chi^j(y,\cdot,t)+\sum_{i=j}^m b_i\partial_tu_i\left(\frac{\tilde{D}^I_{j(j-1)}}{D^I_i}y+\frac{D^I_{j-1}}{D^I_i},t\right)\right]\end{aligned} \tag{69}$$

and

$$\begin{aligned}&\partial_xp^j(x,t)=A_0\tilde{D}^I_{j(j-1)}e^{A_0D^I_{j(j-1)}x}p^{j-1}(1,t)\\&+\tilde{D}^I_{j(j-1)}\left[A_1D^S_0(\bar{\chi}^j(x,\cdot,t))\int_0^1\chi^j(x,\xi,t)d\xi\right.\\&\left.+\sum_{i=1}^{j-1}b_i\kappa^i_0(\chi^j(x,\cdot,t))+\sum_{i=j}^m b_iu_i\left(\frac{\tilde{D}^I_{j(j-1)}}{D^I_i}x+\frac{D^I_{j-1}}{D^I_i},t\right)\right]\\&+\tilde{D}^I_{j(j-1)}\int_0^x A_0\tilde{D}^I_{j(j-1)}e^{A_0\tilde{D}^I_{j(j-1)}(x-y)}\\&\times\left[A_1D^S_0(\bar{\chi}^j(y,\cdot,t))\int_0^1\chi^j(y,\xi,t)d\xi\right.\\&\left.+\sum_{i=1}^{j-1}b_i\kappa^i_0(\chi^j(y,\cdot,t))+\sum_{i=j}^m b_iu_i\left(\frac{\tilde{D}^I_{j(j-1)}}{D^I_i}y+\frac{D^I_{j-1}}{D^I_i},t\right)\right]dy\\&=A_0\tilde{D}^I_{j(j-1)}e^{A_0D^I_{j(j-1)}x}p^{j-1}(1,t)\\&+\tilde{D}^I_{j(j-1)}e^{A_0\tilde{D}^I_{j(j-1)}x}\left[A_1D^S_0(\bar{\chi}^j(0,\cdot,t))\int_0^1\chi^j(0,\xi,t)d\xi\right.\\&\left.+\sum_{i=1}^{j-1}b_i\kappa^i_0(\chi^j(0,\cdot,t))+\sum_{i=j}^m b_ju_j\left(\frac{D^I_{j-1}}{D^I_i},t\right)\right]\\&+\tilde{D}^I_{j(j-1)}\int_0^x e^{A_0\tilde{D}^I_{j(j-1)}(x-y)}\left[A_1\frac{D^S_0}{d\bar{\chi}}\cdot\partial_x\bar{\chi}^j(y,\cdot,t)\right.\\&\times\int_0^1\chi^j(y,\xi,t)d\xi+A_1D^S_0(\bar{\chi}^j(y,\cdot,t))\int_0^1\partial_x\chi^j(y,\xi,t)d\xi\\&+\sum_{i=1}^{j-1}b_i\frac{\kappa^i_0}{d\chi}\cdot\partial_x\chi^j(y,\cdot,t)\\&\left.+\sum_{i=j}^m b_i\frac{\tilde{D}^I_{j(j-1)}}{D^I_i}\partial_xu_i\left(\frac{\tilde{D}^I_{j(j-1)}}{D^I_i}y+\frac{D^I_{j-1}}{D^I_i},t\right)\right]dy\end{aligned} \tag{70, 71}$$

in which the last equation was obtained using an integration by parts. One can observe from (40) that $\bar{\chi}^j(0,\cdot,t) = \bar{\chi}^{j-1}(1,\cdot,t)$. Also, from (62) at order $j-1$ evaluated for $y=1$, one concludes that $\tilde{D}^I_{(j-1)(j-2)}\partial_t p^{j-1}(1,t) = \partial_x p^{j-1}(1,t)$. Using this fact along with (41), one concludes that $\chi^j(0,\cdot,t) = \chi^{j-1}(1,\cdot,t)$. With these considerations and using (64) at order $j-1$, one obtains that

$$\begin{aligned}
&\tilde{D}^I_{j(j-1)}\partial_t p^j(x,t) - \partial_x p^j(x,t) = \tilde{D}^I_{j(j-1)}\int_0^x e^{A_0\tilde{D}^I_{j(j-1)}(x-y)} \\
&\times\Bigg[A_1\frac{dD_0^S}{d\chi}\cdot(\tilde{D}^I_{j(j-1)}\partial_t\bar{\chi}(y,\cdot,t) - \partial_x\bar{\chi}(y,\cdot,t))\int_0^1\chi(y,\xi,t)d\xi \\
&+A_1D_0^S(\bar{\chi}(y,\cdot,t))\int_0^1(\tilde{D}^I_{j(j-1)}\partial_t\chi(y,\xi,t) - \partial_x\chi(y,\xi,t))d\xi \\
&+\sum_{i=1}^{j-1}\frac{d\kappa_i^0}{d\chi}\cdot(\tilde{D}^I_{j(j-1)}\partial_t\chi(y,\xi,t) - \partial_x\chi(y,\xi,t))\Bigg]dy \qquad (72)
\end{aligned}$$

From there, using the same arguments as in [8], one can conclude that (62)–(63) hold at order j. Finally, as $\tilde{D}^I_{j(j-1)}\partial_t p^j(1,t) = \partial_x p^j(1,t)$ from (62) evaluated for $y=1$, using (70), one obtains (64) at order j.

Hence, one concludes that (62)–(64) hold for $j=1,\ldots,m$.

Consequently, from (42), it follows that, for $x \in \left[\frac{D_i^I}{D_j^I}, \frac{\tilde{D}^I_{i+1}}{D_j^I}\right]$ for a given $i=0,\ldots,j-1$,

$$\begin{aligned}
&D_j^I\partial_t w_j(x,t) - \partial_x w_j(x,t) = D_j^I\partial_t u_j(x,t) - \partial_x u_j(x,t) \\
&-\frac{D_j^I}{\tilde{D}^I_{(i+1)i}}\kappa_j^0\left(\tilde{D}^I_{(i+1)i}\partial_t\chi^{i+1}\left(\frac{D_j^I}{\tilde{D}^I_{(i+1)i}}\left(x-\frac{D_i^I}{D_j^I}\right),\cdot,t\right)\right. \\
&\left.-\partial_x\chi^{i+1}\left(\frac{D_j^I}{\tilde{D}^I_{(i+1)i}}\left(x-\frac{D_i^I}{D_j^I}\right),\cdot,t\right)\right) \qquad (73)
\end{aligned}$$

Therefore, using (33) and (62), it follows that $D_j^I\partial_t w_j(x,t) = \partial_x w_j(x,t)$ for all $x\in[0,1]$. This concludes the proof.

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