ALGEBRAS AND NONLINEAR MULTIRESOLUTION ANALYSIS THAT ARE CONSISTENT WITH THE STRANG AND FIX CONDITIONS

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ABSTRACT

We investigate how the concept of multiresolution analysis can be adapted in order to handle nonlinear operators. Under reasonable assumptions this question has a unique answer, which we caracterize and build explicitely here.

1. MOTIVATION

1.1. General background

Given a non linear operator F on a functional space E, and a basis $\mathcal{B}$ of E, we may be interested in the relationship which exists between the coordinates of a signal x and that of Fx in $\mathcal{B}$. After all, this is where the actual computations should take place. For instance, we study the wavelet coefficients of the square of a signal as a function of the wavelet coefficients of the signal (see [3], [1] for classics on wavelets). Two criteria can be loosely used to qualify the fitness of the basis with respect to F: the structure of the coordinates of x should be preserved as much as possible in the coordinates of Fx, and, transversally, the nonlinear structure of F should be preserved as much as possible in its coordinate-to-coordinate version.

Because wavelets provide a nice approximation of a local time/frequency analysis, they appear more qualified than the Fourier transform to analyse nonlinear operators, espacially dynamical systems. Unforunately, they do not satisfy the previous requirements. We study here how to find a more suitable environment for combined signal analysis and nonlinear operations.

Potential applications are all classical nonlinear processes, whether they be static (like 3-D image processing, for instance), or dynamic (like nonlinear input-output systems). They also include the modelization of non classical systems, which could be a mixture of fractals and dynamical and/or nonlinear systems.

1.2. An example where things work out fine

Let $h_k = 1_{k\delta,(k+1)\delta}$ a rescaled Haar basis, and define the associated projection Π_δ by $\Pi_\delta x = \sum_{k\in\mathbb{Z}} c_k h_k$ with $c_k = 1/\delta \int_{\mathbb{R}} x(s)h_k(s)ds$. Now consider a polynomial Q, a signal x, its projection $e = \Pi_\delta x$, and its remainder $w = x - e$. Then we can compute $\Pi_\delta(Q(x))$ using the following formula:

$$(1) \qquad \boxed{\Pi_\delta(Q(x)) = \sum_{k=0}^{k=+\infty} \frac{Q^{(k)}(e)}{k!}\Pi_\delta(w^k)}$$

The interesting fact is that, in this formula, the polynomial is only applied to the projection e, while the projection is only applied to the powers of w. This can be traced back to the following two properties of the Haar basis:

- P1 each resolution is an algebra
- P2 the product of an element of an oscillation (or wavelet) space will an element of a slower resolution is an element of the oscillation space

The last property makes the left product an (almost) block triangular linear operator over the wavelet analysis. Hence the signal's decomposition is rather well preserved through basic (e.g. polynomial) nonlinear operations (diagonalization is out of reach).

Unfortunately, these properties do not generalize to other multiresolution analysis. In particular P1 is always false unless the projection commutes with the product in the image E_δ of Π_δ.

2. STUDY OF PROPERTY P1

In the rest of the paper E_δ will be a subspace of L^2_{loc} spanned by the family $\varphi_{\delta,k}(t) = \varphi(t/\delta - k)$, and (φ, φ^*) will be a biorthogonal system; $\varphi^*_{\delta,k}$ will denote $1/\delta\varphi^*(t/\delta - k)$, Π_δ will be the projection $\Pi_\delta x = \sum_k < x, 1/\delta\varphi^*_{\delta,k} > \varphi_{\delta,k}$.

2.1. Step 1: making E_δ an algebra

We begin with the simpler problem of making *one* resolution stable by product. To achieve this, we shall replace the inadequate classical product law with another one which *approximates it at an optimal order*. We recall the first the Stang and Fix conditions:

Theorem 1 (Strang & Fix [2]) *Given an integer p, the two following properties are equivalent:*

$$(2) \quad \exists C \left(\forall f \in H^{p+1}(\mathbb{R})\right) (\forall \delta \leq 1) \quad \|P_\delta f - f\|_{L^2} \leq C\delta^{p+1} \left\| \frac{d^{p+1} f}{dx^{p+1}} \right\|$$

$$(3) \qquad (\forall p \in \{0, \dots, p\}) \quad P_1(x^p) = x^p$$

Hence, the optimal order for approximating the product is δ^{p+1}. We can caracterize such optimally approximating product laws. To do so, we assume that E_δ have an algebra structure $(E_\delta, +, *_\delta, .)$ for δ close to 0. We also assume that, for x and y in E_0, one has $x\,(t/\delta) *_\delta y\,(t/\delta) = (x *_0 y)\,(t/\delta)$, that there exists $L > 0$ such that $x(t-L) *_0 y(t-L) = (x *_0 y)\,(t-L)$, and that there exists some threshold M such that, for compactly supported elements of E_0, $\mathrm{Dist}(\mathrm{Dom}(x), \mathrm{Dom}(y)) \geq M \Longrightarrow x *_0 y = 0$

Theorem 2 (Approximation of the product) *Let p the order of the Strang and Fix conditions for the space E. The two following conditions are equivalent:*

$$(4) \quad (\exists K, M, N) \left(\forall x, y \in C^{p+1}(\mathbb{R})\right) \sup_{t \in [a,b]} |xy - (\Pi_\delta)\, x *_\delta (\Pi_\delta y)| \leq K\delta^{r+1} \sup_{[a-M, b+M]} \left(\left|x^{(r+1)}\right| + \left|y^{(r+1)}\right| + \left|(xy)^{(r+1)}\right| \right)$$

$$(5) \qquad t^i *_\delta t^j = t^{i+j} \quad \textit{if } i + j \leq p$$

Proof: it is similar to the proof of theorem 1. The necessary condition is obtained by letting $x(t) = t^i$ in (2). The sufficient condition can be drawn from lemma 1 hereafter, and using $P_\delta T_\delta x$ instead of x ■

A similar theorem can be proved for the differentiation, with the difference that one order of approximation has to be dropped.

There remains to find such a product law. To that end, we generalize the Lagrange interpolation theorem in order to identify polynomials from their coordinates in E_δ. We will denote by $\mathbb{R}_p[t]$ the space of polynomials of degree lower than p, $\pi_{\delta,0}$ will denote the projection on $E_{\delta,0} \stackrel{def}{=} \mathrm{span}(\varphi_0, \dots, \varphi_p)$, and F the canonical linear isomorphism between $\mathbb{R}[t]$ and $\mathbb{R}(t)$

Theorem 3 (Polynomial representation) $P_{\delta,0} \stackrel{def}{=} \pi_{\delta,0}$ *is a linear space isomorphism between* $\mathbb{R}_p[t]$ *and* $E_{\delta,0}$.

Proof: Let $T_i = \pi_{\delta,0}(t^i)$, and let us write T_i as $T_i = \sum_{k=0}^{k=p} a_{k,i}\varphi_k$. We have $a_{k,i} = < \varphi_k^*, t^i > = < \varphi_0^*, (t+k)^i >$

Define $A = (a_{i,j})$, and B as the matrix that expresses the basis $(1, (t+1), \dots (t+1)^p)$ of $\mathbb{R}[t]$ in the basis $(1, t, \dots t^p)$, that is, $B = (b_{i,j})$ with $b_{i,j} = \begin{pmatrix} j-1 \\ i-1 \end{pmatrix}$ if $i \leq j$ and $b_{i,j} = 0$ if $i > j$. B is invertible, and the matrix that changes the basis $(1, (t+k), \dots (t+k)^p)$, $k \in \mathbb{Z}$, into $(1, t, \dots t^p)$ is equal to B^k. If we denote by M the vector matrix of the $p+1$ first moments of φ^*, then we have:

$$(6) \qquad A = \begin{bmatrix} M^T & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & M^T \end{bmatrix} \begin{bmatrix} \mathrm{Id} \\ B \\ \vdots \\ B^p \end{bmatrix}$$

Let us now denote by $\mathcal{B}$ a matrix that is obtained from B by substituting each of its element by the outer product of it with the identity matrix that is the size of B. Then we have

$$(7) \qquad \begin{bmatrix} \mathrm{Id} \\ B \\ \vdots \\ B^p \end{bmatrix} = \mathcal{B}^T \begin{bmatrix} \mathrm{Id} \\ B - \mathrm{Id} \\ \vdots \\ (B - \mathrm{Id})^p \end{bmatrix}$$

$B - \mathrm{Id}$ is nihilpotent; more precisely, $(B - \mathrm{Id})^k$ has its first k columns equal to zero (as well as its last k rows). On the other hand, we can see that the product of the large M^T-diagonal matrix by $\mathcal{B}^T$ is in fact equal to $\mathcal{B}^T M^T$, so that we have, in the end:

$$(8) \qquad A = \mathcal{B}^T M^T \begin{bmatrix} \mathrm{Id} \\ B - \mathrm{Id} \\ \vdots \\ (B - \mathrm{Id})^p \end{bmatrix}$$

Considering the structure of $(B - \mathrm{Id})^k$, we see that the product of M^T with the large matrix on the right is upper triangular, with ones on the diagonal. Since $\mathcal{B}$ is regular, this proves that A has full rank ■

Now let us make $\mathbb{R}_p[t]$ an algebra; to do so, we define the product $\times$ over $\mathbb{R}_p[t]$ as the expansion of the usual product at the order p and at point 0. We define the differential D as the usual one. Through $P_{\delta,0}$, this expands $E_{\delta,0}$ into a differential ring $(E_{\delta,0}, +, *_{\delta,0}, D_{\delta,0})$. The latter can be extended to the whole of E_δ by using the shift operator σ_δ of length $(p+1)\delta$. This can be summed up as $(E_{\delta,k}, +, *_{\delta,k}, D_{\delta,k}) = \sigma_{k\delta}(E_{\delta,0}, +, *_{\delta,0}, D_{\delta,0})\sigma_{-k\delta}$.

Then the global ring $(E_\delta, +, *_\delta, D_\delta)$ is identified to the tensorial product $\bigotimes_{k \in \mathbb{Z}} (E_{\delta,k}, +, *_{\delta,k}, D_{\delta,k})$, which is itself defined by isomorphism to $\bigotimes_{k \in \mathbb{Z}} (\mathbb{R}_p[t - k(p+1)\delta], +, *, D)$.

Remark that we can define another piecewise polynomial representation of x using a sequence of Taylor expansion. Let $T_k = k(p+1)$ and denote by $\mathcal{T}_\delta x$ the function with value $\sum_{i=0}^{i=p} (x^{(i)}(\delta T_k))(t - \delta T_k)^i / i!$ over the interval $[\delta T_k, \delta T_{k+1}[$. Then $P_\delta \mathcal{T}_\delta x$ approximates x:

Lemma 1 *there exist two constants C and b such that, for any functions x of class* C^{p+1} *and any real number a,*

$$(9) \quad \sup_{t \in [a, a+1]} |x(t) - P_\delta \mathcal{T}_\delta x(t)| \leq C\delta^{r+1} \sup_{t \in [a-b, a+b+1]} \left|x^{(r+1)}(t)\right|$$

Proof: because of theorem 1, we can replace x with $\Pi_\delta x$ above. Let $\pi_{\delta,k} = \sigma_\delta^k \pi_{\delta,0} \sigma_\delta^{-k}$; then $\Pi_\delta - P_\delta \mathcal{T}_\delta =$

$\sum_{k\in\mathbb{Z}} \pi_{\delta,k}(Id - T_{\delta,k})$. But each $\pi_{\delta,k}(Id - T_{\delta,k})$ is localized, and of order δ^{p+1} ■

Let us show now that $*_\delta$ satisfies the conditions of theorem 2. In fact, we can caracterize the product law $*_\delta$ as follows:

Theorem 4 *Consider a candidate product law $**_\delta$ on E_δ and $\times\times$ a candidate product law on $\mathbb{R}_p[t]$. Both product are assumed to associative, and distibutive over +. The two following properies are equivalent:*

***a)** $**_\delta$ satifies condition (4), and the following diagram commutes:*

$$(10)\quad \begin{array}{ccc} (E_1,+,**_1) & \xrightarrow{P_1} & \underset{k\in\mathbb{Z}}{\otimes}(\mathbb{R}_p[t-T_k],+,\times\times) \\ \mathcal{R}_\delta\downarrow & & \mathcal{R}_\delta\downarrow \\ (E_\delta,+,**_\delta) & \xrightarrow{P_\delta} & \underset{k\in\mathbb{Z}}{\otimes}(\mathbb{R}_p[t-\delta T_k(\delta)],+,\times\times) \\ \sigma_\delta\downarrow & & \sigma_\delta\downarrow \\ (E_\delta,+,**_\delta) & \xrightarrow{P_\delta} & \underset{k\in\mathbb{Z}}{\otimes}(\mathbb{R}_p[t-\delta T_k],+,\times\times) \end{array}$$

where $\mathcal{R}_\delta$ denotes the resaling operator ($(\mathcal{R}_\delta x)(t) = x(\delta t)$)

***b)** $\times\times$ is the Taylor expansion of classical product at the order p, e.g., $**_\delta = *_\delta$.*

Proof: *Necessary condition:* if we take $x(t) = t^i$ and $y(t) = t^j$ in (4), with $i + j \leq p$, then the right handside is 0, and we recover condition (5), e.g., $t^i * *_\delta t^j = t^{i+j}$ for $i + j \leq p$. Now, the product law $\times\times$ is entirely determined by the decomposition $\sum_{i=0}^{i=p} \alpha_i t^i$ of $t^p \times \times t$ along the basis $1, t, \dots t^p$. So, let us consider $x(t) = t^p$ and $y(t) = t$ in (4). Following lemma 1, we shall use $P_\delta T_\delta(xy)$ in place of xy. It has the following piecewise polynomial representation:

$$(11)\quad P_\delta T_\delta(xy)(t) = \sum_{i=0}^{i=p} \binom{p+1}{i} (t-k\Delta)^i (k\Delta)^{p+1-i} \quad \text{in } \mathbb{R}_p[t-\delta T_k]$$

with $\Delta = (p+1)\delta$. Because the polynomials of degree $\leq p$ are unchanged by the various transformations, the expression of $t^p \times \times t$ in $\mathbb{R}_p[t - \delta T_k]$ is $(t - k\Delta + k\Delta)^p \times \times (t - k\Delta + k\Delta)$. After expanding the previous formula, we get

$$(12)\quad \sum_{i=0}^{i=p} \binom{p}{i} (t-k\Delta)^i (k\Delta)^{p+1-i} + \sum_{i=1}^{i=p} \binom{p}{i-1} (t-k\Delta)^i (k\Delta)^{p+1-i} + \sum_{i=0}^{i=p} \alpha_i (t-k\Delta)^i$$

The difference between $P_\delta T_\delta(xy)$ and $x * *_\delta y$ is equal to $\left(\sum_{i=0}^{i=p} \alpha_i (t-k\Delta)^i\right)_{k\in\mathbb{Z}} \stackrel{def}{=} z$. It can be expressed as the rescaling of a δ independant function. More precisely, we have:

$$(13)\quad z = \mathcal{R}_\delta \left\{ \sum_{i=0}^{i=p} \alpha_i \delta^i \left(\pi_{1,k}(t-k(p+1))^i\right)_{k\in\mathbb{Z}} \right\}$$

Assume that there is a non-zero α_i, and the i_0 the smallest index for which $\alpha_i \neq 0$. Then the term with coefficient α_{i_0} dominates the others in $P_1^{-1}z$ when δ tend to 0. Hence, z is of order δ^{i_0}; but on the other hand, it should be locally smaller than som $K\delta^{p+1}$. This implies that all of the α_i are equal to 0, and the result

Sufficient condition:

Commutation: we have to show mainly the polynomial representation commutes with rescaling. Let us consider $x \in E_1$, which we write as $x = \sum_{k\in\mathbb{Z}} \sum_{i=0}^{i=p} x_{i,k} \varphi(t - T_k - i)$, and let $y(t) = x(t/\delta)$. Denote by A_δ the matrix that we used in the proof of theorem 3. We have

$$(14)\quad A_\delta = A_1 \times \left[\underset{1\leq i\leq p+1}{\text{Diag}} \left(\delta^{i-1}\right) \right] \stackrel{def}{=} A_1 D_\delta$$

Let us write now $P_1^{-1}x$ as $\left(\sum_{i=0}^{i=p} \alpha_{i,k}(t-T_k)^i\right)_{k\in\mathbb{Z}}$. Then

$$\begin{aligned} \left[\mathcal{P}_\delta^{-1} y\right](t) &= \left([1, \dots, (t-\delta T_k)^p] D_\delta^{-1} A_0^{-1} [x_{0,k}, \dots, x_{p,k}]^T\right)_{k\in\mathbb{Z}} \\ &= \left(\sum_{i=0}^{i=p} \delta^{-i} \alpha_{i,k} (t-\delta T_k))^i \right)_{k\in\mathbb{Z}} \\ &= \left[\mathcal{P}_0^{-1} x\right] \left(\frac{t}{\delta}\right) \end{aligned}$$

Approximation: this comes from the commutation result, and the generalized Strang and Fix condition of theorem 2 ■

Remark: a similar (though simpler) result exists for the differentiation.

2.2. Step 2: building E_δ into a multiresolution analysis

It follows the lines of the classical theory, except that the rescaling is done towards the larger steps, and that *the rescaling is performed in the piecewise polynomial representation*, then tranported to E_δ. This is because the nonlinear structure is truly defined in $\otimes\mathbb{R}_p[t - \delta T_k]$. The reader will check that the two rescaling *are* different. If we denote by $\mathcal{P}$ the functional image (as piecewise polynomial functions with step $(p+1)\delta$) of $\otimes\mathbb{R}_p[t - T_k]$, then we essentially manipulate the subspaces

$$(15)\quad \mathcal{P}_j = \left\{p(2^{-j}t),\ p \in \mathcal{P}\right\} \quad j \geq 0$$

2.3. Step 3: checking property P1

Unfortunaly, property P1 does not extend to the case $j > 0$, except for the Haar basis. It is a consequence of the following theorem:

Theorem 5 *Let $\mathcal{Q}$ a subalgebra of $\mathcal{P}$ using the previous Taylor expansion product law, and D its domain, that is, the union of the domains of all elements of $\mathcal{Q}$. Assume that $\mathcal{Q}$ includes the restrictions to D of the polynomial functions of degree 0 and 1.*

Then $\mathcal{Q}$ is the restriction of $\mathcal{P}$ to D.

Proof: because of the structure of $\mathcal{P}$, we can represent its domain by the sequence of indices k_i such that $[\delta T_{k_i}, \delta T_{k_i+1}] \subset D$. The T_{k_i} increase with i.

We are going to show that, for any N and M, $\mathcal{Q}$ includes the restriction of $\mathcal{P}$ to $D_1 \stackrel{def}{=} \bigcup_{i=N}^{i=N+M-1}[\delta T_{k_i}, \delta T_{k_i+1}]$. This will show that $\mathcal{Q}$ is algebraically equal to the restriction of $\mathcal{P}$ to the domain of $\mathcal{Q}$.

Since $\mathcal{Q}$ is stable by product, it includes the restrictions to D of the monomials $\theta^k \stackrel{def}{=} (\underbrace{t * \ldots * t}_{k \text{ times}})$. Now let us consider a sequence of a_i such that $\sum_{i=0}^{i=M(p+1)-1} a_i\theta^i = 0$ over D_1; and let us define $(p_k)_{k\in\mathbb{Z}}$ as the representation of the previous in $\otimes\mathbb{R}_p[t - \delta T_{i_k}]$. If we use the classical differentiation[1] on the linear space $\mathbb{R}_p[t - \delta T_{i_k}]$, then we check that, for $0 \leq j \leq p$, $p_k^{(j)}(0)$ is equal to the j^{th} derivative of the classical polynomial $p(t) = \sum_{i=0}^{i=M(p+1)-1} a_i t^i$ at the point δT_{i_k}. All these values are zero; hence $p = 0$, and the a_i are zero. The restriction of $\mathcal{Q}$ to D_1 is then of maximal dimension $(p+1)M$, and equal to the restriction of $\mathcal{P}$ to D_1 ■

Though property P1 does not seem to extend beyong the Haar basis, property P2 has an interesting extension.

3. STUDY OF PROPERT P2 FOR THE TAYLOR PRODUCT

We consider the product law $*_\delta$ as caracterized in theorem 4. A necessary and sufficient condition for property P2 to hold is the existence of an *absorbing innovation space*, as described below:

Definition 1 *A subspace $\mathcal{Q}_1$ of $\mathcal{P}_0$ is an absorbing innovation subspace of $\mathcal{P}_0$ if the three following properties are satisfied*

- $\mathcal{P}_0 = \mathcal{P}_1 \oplus \mathcal{Q}_1$.
- *the product of an element of $\mathcal{P}_1$ and of an element of $\mathcal{Q}_1$ is an element of $\mathcal{Q}_1$.*
- *$\mathcal{Q}_1$ in invariant under the action of the shifts of length $2(p+1)\delta$*

If we define $\mathcal{Q}_{\delta,j} = \{q(2^{-j+1}t),\ q \in \mathcal{Q}_1\}$, then $\mathcal{P}_0 = \mathcal{P}_j \oplus_{i=1}^{i=j} \mathcal{Q}_i$ and the second property is actually equivalent to P2. We can give two examples of generic absorbing innovation spaces:

Theorem 6 *Let $\mathcal{P}_{0,k}$ the space of elements of $\mathcal{P}_0$ with a domain in the interval $[\delta T_k, \delta T_{k+1}]$. Define $\mathcal{Q}_1^c$ and $\mathcal{Q}_1^{ac}$ as*

- $\mathcal{Q}_1^c = \bigoplus_{k\in\mathbb{Z}} \mathcal{P}_{0,2k+1}$
- $\mathcal{Q}_1^{ac} = \bigoplus_{k\in\mathbb{Z}} \mathcal{P}_{0,2k}$

Then $\mathcal{Q}_1^c$ and $\mathcal{Q}_1^{ac}$ are absorbing innovation spaces for all tensorial product laws on $\mathcal{P}_0$.

It turns out that these are the only possible absorbing innovation spaces when using the Taylor expansion of the product.

Lemma 2 *The assumptions are the same as in theorem 5. Then $\mathcal{Q}_1$ is an ideal of $\mathcal{P}$.*

[1] that is, we do not take the "jumps" into account

Proof: because $\mathcal{Q}_1$ is absorbing over $\mathcal{P}_1$, it is absorbing over the algebra generated by $\mathcal{P}_1$. Because of theorem 5, this algebra is equal to $\mathcal{P}$. This implies that $\mathcal{Q}_1$ is an ideal of $\mathcal{P}$ ■

We consider here that the product law on $\mathcal{P}_0$ is $**_\delta$, e.g., the Taylor expansion of the classical product up to degree p, or, equivalently, the functional image of $*_\delta$.

Theorem 7 *$\mathcal{Q}_1^c$ and $\mathcal{Q}_1^{ac}$ are the only absorbing innovation subspaces of $(\mathcal{P}_0, +, **)$.*

Proof: For a non zero polynomial $p(t)$, $val(p)$ will denote the lowest power of t which has a non zero coefficient. The innovation subspace $\mathcal{Q}_1$ is entirely determined by its subspace $\mathcal{Q}_{-1,0}$ of its elements with zero value outside of $(0, \delta T_2)$. It is of dimension $p+1$. Let us consider an element q of $\mathcal{Q}_{-1,0}$ with value $q_1(t)$ (resp. $q_2(t - \delta T_1)$) on $(0, \delta T_1)$ (resp. $(\delta T_1, \delta T_2)$). Such an element exists becauses of the absorbing property; moreover q_1 or q_2 must have a non zero constant because $\mathcal{P}_0 = \mathcal{Q}_1 \oplus \mathcal{P}_1$. Let us assume that it is p_1. Then q_1 is invertible in $(\mathcal{P}_0, **_\delta)$ (use the Taylor expansion of the classical inverse). Hence we can assume that $p_1 = 1$. We can see that the family $(q, q**_\delta t, q**_\delta t^2, \ldots, q**_\delta t^p)$ is a basis of $\mathcal{Q}_{-1,0}$. Now assume that $q_2 \neq 0$. Then we can see that $val(q_2 *_\delta (t + \delta T_1)^i)$ does not depend on i. This implies that $q**_\delta t^{p+1}$ is not zero. But it has a zero value on $(0, \delta T_1)$. By computing its ccordinates in the basis $(q, q**_\delta t, q**_\delta t^2, \ldots, q**_\delta t^p)$, we see on the left interval that these coordinates should be 0, and hence, that $p**_\delta t^{p+1} = 0$. This contradiction shows that q_2 is necessarily 0, and that $\mathcal{Q}_1 = \mathcal{Q}_1^{ac}$. $\mathcal{Q}_1^c$ is obtained by assuming that q_2 has a non zero constant instead ■

Corollary 1 *Let $\mathcal{S}_j = \oplus_{i=1}^{i=j}\mathcal{Q}_j$. Then*

- *either $\mathcal{S}_j = \{x \in \mathcal{P}_0$ s.t. $x(t) = 0$ over $(\delta T_{k2^{-j}}, \delta T_{k2^{-j}+1})$, $k \in \mathbb{Z}\}$ if $\mathcal{Q}_1 = \mathcal{Q}_1^{ac}$*
- *or $\mathcal{S}_j = \{x \in \mathcal{P}_0$ s.t. $x(t) = 0$ over $(\delta T_{k2^{-j}-1}, \delta T_{k2^{-j}})$, $k \in \mathbb{Z}\}$ if $\mathcal{Q}_1 = \mathcal{Q}_1^c$*

and $\mathcal{S}_j$ is an algebra.

Proof: Left to the reader ■

Now let us turn to the equivalent of formula (1).

Corollary 2 (Commutation formula) *Let $\mathbb{P}_j$ the operator on $\mathcal{P}_0$ defined by setting to zero the components in $\mathcal{S}_j$ in the decomposition $\mathcal{P}_0 = \mathcal{P}_j \oplus \mathcal{S}_j$.*

Let Q a polynomial and x an element of $\mathcal{P}$, $e = \mathbb{P}_j x$ and $w = x - e$. Then

$$\boxed{\mathbb{P}_j[Q(x)] = \mathbb{P}_j[Q(e)]} \qquad (16)$$

4. REFERENCES

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