Boundary Estimation of Boundary Parameters for Linear Hyperbolic PDEs

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Abstract—We propose an adaptive observer scheme to estimate boundary parameters in first-order hyperbolic systems of Partial Differential Equations (PDE). The considered systems feature an arbitrary number of states travelling in one direction and one counter-convecting state. Uncertainties in the boundary reflection coefficients and boundary additive errors are estimated relying on a pre-existing observer design and a novel Lyapunov-based adaptation law.

Index Terms—Kalman filtering, partial differential equations (PDEs).

I. INTRODUCTION

SYSTEMS of first-order hyperbolic Partial Differential Equations (PDEs) are predominant in numerous applications involving transport phenomena, e.g., traffic control [15], open-channel flow [3] or oil production and drilling [13], [6]. Many of these applications feature uncertainties in model parameters that prompt the need for adaptive observer and control design. Certain oil well drilling techniques, referred to as UnderBalanced Drilling, consist in producing oil and gas while drilling, resulting in a multiphase flow inside the well. The dynamic state of this flow, which satisfies a system of first-order hyperbolic PDEs, must be monitored to ensure efficiency and safety of the operation. However, only boundary measurements are typically available in such operations, and the physical properties of the reservoir being drilled are generally poorly known. Estimating boundary (reservoir) parameters is of interest, in this case, not only to improve monitoring of the process but also to collect information on the future productivity of fields.

Typical approaches consist in applying classical estimation approaches, e.g., Kalman filtering [17] or particle filtering [21] to simultaneously estimate states and parameters. These methods, however, have no guarantee of convergence and provide little insight into the observability or identifiability of such systems. In turn, estimation and control schemes for hyperbolic systems have, in a substantial part, been enabled by the design of Lyapunov functions [3], [4] and the backstepping method [11], [20]. Lyapunov functions provide means to study the convergence of the proposed algorithms, while backstepping is a powerful design tool that has widely been used in the context of PDEs (see, e.g., [1], [19]). Another advantage of Lyapunov functions is to enable the design of adaptive schemes, as has been done in the case of parabolic PDEs [12], wave equations [10] or delay systems [2].

In this paper, we consider systems of linear hyperbolic PDEs featuring an arbitrary number of states with positive transport speeds and one state with a negative transport speed. The motivation behind studying such systems lies in the above-mentioned drilling application for which certain multiphase flow models have this particularity [16].

In [5], an adaptive observer is presented for this class of systems, able to estimate uncertain additive errors at one boundary. The observer design uses a measurement at the boundary where the uncertainty is located, while the adaptive law uses unmatched measurements. However, this method could not straightforwardly be extended to estimate multiplicative errors (i.e., uncertainty in reflection coefficients). Similarly, in [1] and [7], additive time-varying disturbances are estimated using boundary measurements. Again, the method presented there does not extend to multiplicative uncertainty. Multiplicative uncertainties appear in the aforementioned reservoir characterization problem during drilling, where they result from uncertainty in the reservoir pressure and permeability. In turn, in [8], a single-boundary backstepping observer is designed for hyperbolic systems regardless of the number of states travelling in either direction. Based on this design, we propose here to combine an observer relying on the unmatched measurement with an adaptation law for additive and multiplicative boundary parameters using both boundary measurements. Asymptotic convergence of both state and parameter estimation errors to zero is proved using a Lyapunov function.

The paper is organized as follows. In Section II we define the main objective and briefly recall former results. In Section III we present the main adaptive observer result. In Section V we present simulation results and give some implementation remarks. Most of the technical proofs and developments are contained in Appendices B and C.

1 Namely drift-flux models.
II. PROBLEM DESCRIPTION AND FORMER RESULTS

A. Problem Description

We consider a class of first-order linear hyperbolic systems on the spatial domain \( x \in [0, 1] \) featuring \( n \in \mathbb{N} \) transport equations converging left to right coupled with one transport equation converging right to left. More precisely we consider the following \((n+1)\) equations of the form

\[
\begin{align*}
&u_t(x,t) + \Lambda u_x(x,t) = \Sigma^{++} u(x,t) + \Sigma^{+-} v(t,x) \\
&v_t(x,t) - \mu v_x(x,t) = \Sigma^{-+} u(x,t) + \sigma^{-} v(x,t)
\end{align*}
\]

subject to the boundary conditions

\[
u(t,0) = qv(t,0) + \theta, \quad v(t,1) = ru(t,1) + U(t)
\]

where \( U(t) \) is the control input, \( q, \theta \in \mathbb{R}^{n} \) are uncertain constant vectors of boundary parameters and the model coefficients have the following structure

\[
\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ \vdots & \ddots \end{pmatrix}, \quad \mu \in \mathbb{R}_+
\]

with

\[-\mu < 0 < \lambda_1 < \ldots < \lambda_n\]

and

\[
\Sigma^{+-} = \begin{pmatrix} \sigma_{1}^{+} & \ldots & \sigma_{n}^{+} \end{pmatrix}, \quad \Sigma^{-+} = \begin{pmatrix} \sigma_{1}^{-} & \ldots & \sigma_{n}^{-} \end{pmatrix}, \quad \sigma^{-} \in \mathbb{R}
\]

\[
q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}, \quad r = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}
\]

The existence and uniqueness of a solution \( w \in C([0, \infty); H^1((0,1], \mathbb{R}^{n+n+m})) \) to (1)–(3) is assessed, e.g. in [18, Theorem 3.1]. Moreover, we consider here for simplicity classical \( C^1 \) solutions, which imposes certain compatibility conditions on the initial condition (see e.g. [14]).

Remark 1: Constraint \((4)\) imposes that the system is strictly hyperbolic. This is a necessary condition for the design of a backstepping observer, as illustrated, e.g. in [8] and [9] and as will appear in Equation (A.70).

Remark 2: We consider here constant coupling coefficients and transport velocities for the sake of readability, however the method straightforwardly extends to spatially varying coefficients with more involved technical developments.

In what follows, we solve the two following problems.

\textbf{Problem 1:} Assuming that the following measurement vector is available

\[
y(t) = \begin{pmatrix} u(t,1)^T \\ v(t,0) \end{pmatrix}
\]

estimate the states of system (1)–(3) and the uncertain parameters \( q \) and \( \theta \).

Even if Problem 2 is clearly an extension of Problem 1, we treat the two problems separately. This because estimating at the same time \( q \) and \( \theta \) introduces further technical complications, which make the notation heavier and which are unnecessary when only \( q \) is to be estimated. Moreover solving before Problem 1 provides a softer way to introduce the reader to the more involved Problem 2.

B. Former Results

In [5], an adaptive observer is designed to estimate \( \theta \) in (3), assuming \( q \) perfectly known. The design, there, relies on a matched observer, i.e. an observer relying on the measurement \( v(t,0) \).

Then, an adaptation law for the estimated parameter \( \hat{\theta} \) is built by computing the transition matrix between the error \( \theta \) and the measurement \( u(t,1) \), and then performing a gradient descent. This design does not, however, straightforwardly extend to estimating \( q \), since the observer relying on the measurement \( v(t,0) \) is potentially destabilized by the introduction of an error.

In [8], an observer is designed for general heterodirectional linear hyperbolic systems, i.e. featuring arbitrary numbers of states travelling in each direction. In the context of this paper, this enables the design of an unmatched state observer, i.e. relying on the measurement of \( u(t,1) \).

What follows, we rely on such a design and a Lyapunov-based adaptive law using both measurements \( u(t,1) \) and \( v(t,0) \) to simultaneously estimate the states and uncertain parameters.

III. ADAPTIVE OBSERVER DESIGN FOR PROBLEM 1

In this section we propose the adaptive observer for system (1)–(3) and for the unknown parameters \( q \). In what follow, we omit the argument \( (t,x) \) when not strictly necessary.

A. Observer and Adaptation Law

Let \( \bar{q} \in \mathbb{R}^n \) be the any initial guess on \( q \), which is to remain fixed. We define the quantity \( \delta q \) to be the estimate of the uncertainty \( \delta q = q - \bar{q} \). Following [8], we design an observer of the form

\[
\begin{align*}
\hat{u}_t + \Lambda \hat{u}_x &= \Sigma^{++} \hat{u} + \Sigma^{+-} \hat{v} + P^+(x) \hat{u}(t,1) \\
\hat{v}_t - \mu \hat{v}_x &= \Sigma^{-+} \hat{u} + \sigma^{-} \hat{v} + P^-(x) \hat{u}(t,1)
\end{align*}
\]

subject to the following boundary conditions

\[
\begin{align*}
\hat{u}(t,0) &= \bar{q} \hat{v}(t,0) + \hat{\delta q}(t) v(t,0) \\
\hat{v}(t,1) &= ru(t,1) + U(t)
\end{align*}
\]

with the following update law for \( \hat{\delta q} \), \( i = 1, \ldots, n \)

\[
\frac{d}{dt} \hat{\delta q}_i(t) = k_i \text{sign} \left( v(t - \lambda_i^{-1},0) \right) \left( \exp \left( -\frac{\sigma^{++}_{ii}}{\lambda_i} \right) \hat{u}_i(t,1) + \hat{\delta q}_i(t) v(t - \lambda_i^{-1},0) - k_i \hat{\delta q}_i(t) \right) |v(t - \lambda_i^{-1},0)|
\]
with initial condition
\[ \delta q_i(0) = 0 \]  
(10)
where \( \tilde{u}(t, 1) = u(t, 1) - \hat{u}(t, 1), k_i \) for \( i = 1, \ldots, n \) are design parameters and the observer gains \( P^+: [0, 1] \rightarrow M_{n \times n}(\mathbb{R}) \) and \( P^-: [0, 1] \rightarrow M_{1 \times n}(\mathbb{R}) \) are \( \mathcal{L}^\infty \) functions chosen according to the design presented in [8] in the case of perfectly known parameters.\(^2\) Defining the distributed errors \( \tilde{u}(t, x) = u(t, x) - \hat{u}(t, x) \) and \( \tilde{v}(t, x) = v(t, x) - \hat{v}(t, x) \), the observer (5)–(8) leads to the following error system
\[ \begin{align*}
\tilde{u}_t + \Lambda \tilde{u}_x &= \Sigma_{\delta}^{++} \tilde{u} + \Sigma_{\delta}^{+-} \tilde{v} - P^+(x) \tilde{u}(t, 1) \\
\tilde{v}_t - \mu \tilde{v}_x &= \Sigma_{\delta}^{-+} \tilde{u} + \sigma \tilde{v} - P^-(x) \tilde{u}(t, 1)
\end{align*} \]
(11)
with boundary conditions
\[ \begin{align*}
\tilde{u}(t, 0) &= \delta q(t, 0) + \delta q(t) v(t, 0) - \delta q(t) v(t, 0) \\
\hat{v}(t, 1) &= 0
\end{align*} \]
(13)
(14)
Remark 3: One should notice that we do not directly estimate the unknown parameter \( q \), but the uncertainty \( \delta q \). This because having a time-varying estimate \( \hat{q}(t) \) of \( q \) would lead to kernel equations for \( M(x, \xi) \) and \( N(x, \xi) \) with time-varying boundary conditions (note indeed that \( \hat{q} \) enters in the boundary condition (A.72). Consequently, this would lead to time-varying observer gains \( P^+(x) \) and \( P^-(x) \). Moreover this design gives us an additional degree of freedom in the choice of \( \hat{q} \), which is briefly discussed in Section V-A.

Remark 4: Note that the first term of (9), namely the term
\[ k_i \exp \left( -\frac{\sigma t^+}{\lambda_i} \right) \text{sign}(v(t - \lambda_i^{-1}, 0)) \tilde{u}_i(t, 1), \]
(15)
makes the right-hand side of (9) not continuous in \( t \), since, in general, \( \tilde{u}_i(t, 1) \) might not be proportional to \( v(t - \lambda_i^{-1}, 0) \). As a consequence, by directly using (9), classical solutions to (5)–(8) might not necessarily exist. Nevertheless, the potential discontinuities are concentrated in the interval \([0, t_c]\), with \( t_c := \mu^{-1} + \lambda_i^{-1} \). In fact, after \( t_j := \mu^{-1} \), the state \( \beta \) of (23)–(26) is identically null, and consequently, from (25) and from (A.74)–(A.75), one has that for all \( t \geq t_1, \Lambda \tilde{u}_i(t, 0) = (\delta q_1 - \delta q_1(t)) v(t, 0) \). Therefore, from Lemma 3, one obtains that
\[ \tilde{u}_i(t, 1) = \exp(\sigma t^+ / \lambda_i - \delta q_1(t)) v(t - \lambda_i^{-1}, 0) \]
holds for all \( t \geq t_1 \) and thus (9), with \( \hat{q} = 1 \), is continuous for all \( t \geq t_1 \). Moreover, equation (25) and Lemma 3 yield
\[ \tilde{u}_2(t, 1) = \exp \left( -\frac{\sigma t^+}{\lambda_2} \right) (\delta q_2(t) v(t - \lambda_2^{-1}, 0) + \int_0^t h_1(\xi) \delta \hat{\alpha_1}(t - \lambda_2^{-1}, \xi) d\xi), \]
(16)
and, thus, starting from \( t_c = t_3 + \lambda_1^{-1} \), one has that \( \Lambda \tilde{u}_i(t - \lambda_2^{-1}, \xi) \) is proportional to \( v(t - \lambda_2^{-1}, 0) \). As a consequence, for all \( t \geq t_c \), (9) is continuous also for \( i = 2 \). The same reasoning can be extended by induction to all \( i = 3, \ldots, n \), obtaining that, for any \( i = 1, \ldots, n \), (9) is continuous for all \( t \geq t_c \). As a consequence, classical solutions of (5)–(8) can be obtained by scheduling the activation of the update law (9) at any time instant bigger than \( t_c \). The scheduling of the update laws makes sense also for structural reasons, and a particular choice is proposed in Section V-A.

B. Existence of Observer Gains
For completeness purposes, we briefly recall the following result from [8] regarding the non-adaptive observer design.

Lemma 1: Assume \( q \) perfectly known, i.e. \( \delta q = \delta q = 0 \). There exists functions \( P^+(x) \) and \( P^-(x) \) such that the error system (11)–(14) converges in finite time \( T_0 = \mu^{-1} + \sum_{i=1}^n \lambda_i^{-1} \) to zero.

Sketch of the Proof: The gains are constructed by performing the following backstepping transformation
\[ T : (L^2([0, 1], \mathbb{R}))^{n+1} \rightarrow (L^2([0, 1], \mathbb{R}))^{n+1} \]
\[ (\tilde{u}, \tilde{v}) \mapsto (\tilde{\alpha}, \tilde{\beta}) \]
with
\[ \begin{align*}
\tilde{\alpha}_i(t, x) &= \tilde{u}_i(t, x) - \int_x^1 M(x, \xi) \tilde{\alpha}(t, \xi) d\xi \\
\tilde{\beta}_i(t, x) &= \tilde{v}_i(t, x) - \int_x^1 N(x, \xi) \tilde{\beta}(t, \xi) d\xi
\end{align*} \]
(15)
(16)
The kernels \( M : T \rightarrow M_{n \times n}(\mathbb{R}) \) and \( N : T \rightarrow M_{1 \times n}(\mathbb{R}) \) are defined on the upper triangular domain \( T = \{ (x, \xi) | 0 \leq x \leq \xi \leq 1 \} \). If they satisfy equations (A.68)–(A.73), then the transformation (15)–(16) maps (11)–(14) into the following target system
\[ \begin{align*}
\tilde{\alpha}_i(t, x) + \Lambda \tilde{\alpha}_i(x) &= \Sigma \tilde{\alpha}_i(t, x) + \Sigma^{\lambda} \tilde{\beta}_i(x) \\
&- \int_x^1 D^+(x, \xi) \tilde{\beta}(\xi) d\xi \\
\tilde{\beta}_i(t, x) &= -\mu \tilde{\beta}_i(x) + \Sigma^{(-\lambda)} \tilde{\beta}_i(x) - \int_x^1 d^-(x, \xi) \tilde{\beta}(\xi) d\xi
\end{align*} \]
(17)
(18)
with boundary conditions
\[ \begin{align*}
\tilde{\alpha}_i(0, 0) &= q \tilde{\beta}_i(0, 0) + H(\xi) \tilde{\alpha}_i(\xi) d\xi \\
\tilde{\beta}_i(t, 1) &= 0
\end{align*} \]
(19)
(20)
where \( \Sigma \) is defined as
\[ \Sigma = \begin{pmatrix}
\sigma_{11}^{++} & 0 \\
\vdots & \ddots \\
0 & \cdots & \sigma_{nn}^{++}
\end{pmatrix} \]
and the expressions of functions \( D^+ : T \rightarrow \mathbb{R}^n, d^- : T \rightarrow \mathbb{R} \) and \( H : [0, 1] \rightarrow M_{n \times n}(\mathbb{R}) \) are given in Appendix A. A proof of exponential stability of the target and original systems is given in [8]. The observer gains are then given by
\[ \begin{align*}
P^+(x) &= M(x, 1) \Lambda \\
P^-(x) &= N(x, 1) \Lambda
\end{align*} \]
(21)
(22)
where, again, \( M \) and \( N \) are given by (A.68)–(A.73).

\(^2\) i.e. the ones obtained by taking \( \delta q = 0 \) in (7) and applying [8, Section IV].
C. Target System in the Adaptive Case

The following Lemma gives the expression of the target system in the adaptive case.

Lemma 2: Using the same kernels $M$ and $N$ as in the non-adaptive case, transformation (15)–(16) maps system (11)–(14) into the following target system

\[ \dot{\alpha}_t + \Lambda \dot{\alpha}_x = \Sigma \dot{\alpha}_x + \Sigma^{-} \tilde{\beta} - \int_x^1 D^+ (x, \xi) \tilde{\beta} (\xi) d\xi \]

\[ \dot{\beta}_t - \mu \tilde{\beta}_x = \sigma^{-} \tilde{\beta} - \int_x^1 D^- (x, \xi) \tilde{\beta} (\xi) d\xi \]

with boundary conditions

\[ \dot{\alpha}(t, 0) = \tilde{q} \dot{\beta}(t, 0) + \delta q v(t, 0) - \delta q(t) v(t, 0) \]

\[ + \int_0^1 H(\xi) \dot{\alpha}(\xi) d\xi \]

\[ \dot{\beta}(1, t) = 0 \]

where $\Sigma$ is defined as the diagonal of $\Sigma^+$ and the expressions of functions $D^+ : T \to \mathbb{R}^n$, $D^- : T \to \mathbb{R}$ and $H : [0, 1] \to \mathcal{M}_{n \times n} (\mathbb{R})$ are given in Appendix A.

Proof: Equations (23), (24) and (26) are identical to those derived in the case of known $q$, since their derivation does not depend on $q$ but only on $\tilde{q}$. The only equation changed is the boundary condition of $\tilde{\alpha}$. Substituting (15)–(16) in (13) indeed yields

\[ \hat{u}(t, 0) - \hat{v}(t, 0) = \delta q v(t, 0) - \delta q(t) v(t, 0) \]

\[ = \hat{\alpha}(t, 0) + \int_0^1 M(0, \xi) \hat{\alpha}(\xi) d\xi \]

\[ \tilde{q} \dot{\beta}(t, 0) = \int_0^1 \tilde{q} N(0, \xi) \hat{\alpha}(\xi) d\xi \]

From equations (A.74)–(A.75) and (A.72) we obtain

\[ \int_0^1 (M(0, \xi) - \tilde{q} N(0, \xi)) \hat{\alpha}(\xi) d\xi = - \int_0^1 H(\xi) \hat{\alpha}(\xi) d\xi \]

Thus substituting (28) into (27) yields (25).

D. Main Result

We can now state the main result of the paper.

Theorem 1: Consider the error system (11)–(14) together with the update law (9) for $\delta q(t)$, then if $v(t, 0)$ is bounded and it satisfies the following persistent excitation condition

\[ \exists b, \bar{b}, T > 0 \text{ s.t. } \forall t \in [0, \infty) \]

\[ b < \int_t^{t+T} |v(\tau, 0)| d\tau < \bar{b} \]

then (11)–(14) and the estimation error $\delta q - \tilde{\delta} q(t)$ asymptotically converge to zero in $L_1$ norm. More precisely, defining, for any $i = 1, \ldots, n$

\[ V_i(t) = W_i(t) + \Omega_i(t) + s_i |\delta q_i - q_i(t)| \]

where

\[ W_i(t) = p_i \int_0^1 e^{-\delta x} |\delta q_i(t)| dx + w_i \int_0^1 e^{\delta x} |\tilde{\beta}(t)| dx + r_i \int_0^1 e^{\delta x} |\tilde{\beta}(t - \lambda_i^{-1}(1-x))| dx \]

and

\[ \Omega_i(t) = \sum_{k=1}^{i-1} c_{i,k} \delta q_i(t) \]

with appropriate design coefficients $p_i, w_i, r_i, \delta_i, \gamma_i, s_i, c_{i,k}$, $d_{i,k} > 0$, for all $i, k = 1, \ldots, n$ satisfying

\[ p_i > 0 \]

\[ s_i > \max \left\{ \frac{\lambda_i p_i b}{b_i}, 1 \right\} \]

\[ w_i \geq \frac{\lambda_i Q M p_i}{\mu} \]

\[ r_i \geq \frac{\lambda_i q M}{\mu} \]

\[ \gamma_i \geq \frac{\sigma + D}{r_i \lambda_i \mu} (r_i \lambda_i + s_i k_i e^\mu) \]

\[ \delta_i \geq \max \left\{ \frac{\sigma + D}{\lambda_i}, \frac{(p_i + w_i)(\sigma + D)}{w_i \mu} \right\} \]

\[ c_{i,k} \geq \frac{H \lambda_i p_i e^{\mu}}{a_i} \]

\[ d_{i,k} \geq \frac{H s_i k_i e^{\mu}}{a_i} \]

where the $k_i > 0, i = 1, \ldots, n$, appearing in (9) are arbitrary and

\[ \sigma = \max \left\{ \max_{1 \leq i \leq n} |\sigma_i^+|, \max_{1 \leq i \leq n} |\sigma_i^-|, \max_{1 \leq i,j \leq n} |\sigma_{ij}^+|, |\sigma_{ij}^-| \right\} \]

\[ D = \max \left\{ \sup_{1 \leq i \leq n} \sup_{(x, \xi) \in T} |d_i^+(x, \xi)|, \sup_{(x, \xi) \in T} |d_i^-(x, \xi)| \right\} \]

\[ \bar{Q} = \max_{1 \leq i \leq n} |\bar{q}_i| \]

\[ H = \max_{1 \leq i,j \leq n} \left\{ \sup_{x \in [0, 1]} |h_{ij}(x)| \right\} \]

are bounds on the system coefficients. Then, one has

\[ \forall i = 1, \ldots, n \lim_{t \to +\infty} V_i(t) = 0. \]

The proof of Theorem 1 involves lengthy and tedious computations: for readability purposes, we first give here the intuition behind the choice of the update law (9). Consider the target system (23), (26). The $\beta$ state converges to zero after $t = \mu^{-1}$. 


Then, for \( t \geq \mu^{-1} + \lambda_1^{-1} \), one has

\[
\tilde{u}_1(t, 1) = \tilde{\alpha}_1(t, 1) = \exp\left(\frac{\sigma_{11}^+}{\lambda_1}\right) \tilde{\alpha}(t - \lambda_1^{-1}, 0) = \exp\left(\frac{\sigma_{11}^+}{\lambda_1}\right) (\delta q_1 v(t - \lambda_1^{-1}, 0) - \delta q_1 (t - \lambda_1^{-1})v(t - \lambda_1^{-1}, 0))
\]

which yields the following expression for \( \delta q_1 \)

\[
\delta q_1 = \exp\left(\frac{-\sigma_{11}^+}{\lambda_1}\right) \frac{\tilde{u}_1(t, 1) + \dot{\delta q}_1 (t - \lambda_1^{-1})v(t - \lambda_1^{-1}, 0)}{v(t - \lambda_1^{-1}, 0)}
\]  

(43)

The update law is then designed as the following modified first-order filter

\[
\frac{d}{dt} \delta q_1(t) = k_i (\delta q_1(t) - \dot{\delta q}_1(t)) |v(t - \lambda_1^{-1}, 0)|
\]  

(44)

Plugging (43) in (44) yields (9). Multiplying the first-order filter by \( |v(t - \lambda_1^{-1}, 0)| \) ensures that the parameter is no longer updated when excitation is insufficient. Finally, when \( \dot{\delta q}_1 \) has converged to \( \delta q_1 \), the same reasoning is used to adapt \( \delta q_2 \) and the other parameters by induction. We now give a more formal proof.

\section{Proof of Theorem 1}

The proof relies on the following three Lemmas. Lemma 3 establishes a relation among the measurable boundary signal \( \tilde{u}(t, 1) \) and the effect of the uncertainty \( \dot{\delta q} \) on the system.

\textbf{Lemma 3:} For all \( i = 1, \ldots, n \), we define the following time delay operator

\[
\tau_i(t, x) = t - \lambda_i^{-1}(1 - x)
\]

where we omit the time argument for readability when obvious, i.e. \( \tau_i(x) = \tau_i(t, x) \). Consider the error system (11)–(14) and its target system (23)–(26), then for all \( i = 1, \ldots, n \), one has

\[
\tilde{u}_i(t, 1) = \exp\left(\frac{\sigma_{ii}^+}{\lambda_i}\right) \tilde{\alpha}_i(t, 1) = \exp\left(\frac{\sigma_{ii}^+}{\lambda_i}\right) \tilde{\alpha}(t - \lambda_i^{-1}, 0) = \exp\left(\frac{\sigma_{ii}^+}{\lambda_i}\right) (\delta q_i v(t - \lambda_i^{-1}, 0) - \delta q_i (t - \lambda_i^{-1})v(t - \lambda_i^{-1}, 0))
\]

(28)

\[
\tilde{u}_i(t, 1) = e^{-\frac{\sigma_{ii}^+}{\lambda_i}(t - \tau_i(t, 1))} (\delta q_i v(t, 0) - \delta q_i (t - \lambda_i^{-1})v(t - \lambda_i^{-1}, 0))
\]

\[
\tilde{u}_i(t, 1) = \tilde{q}_i \tilde{\beta}(\tau_i(t, 0), 0) + \int_0^{1 - \nu_i} \sum_{k=1}^{m_i} \gamma_k(x, \xi) \beta(\tau_k(t, 0), \xi) d\xi
\]

\[
\tilde{u}_i(t, 1) = \frac{1}{\lambda_i} \int_0^1 e^{-\frac{\sigma_{ii}^+}{\lambda_i}(t - x)} \left[ \delta q_i (t - \lambda_i^{-1})v(t - \lambda_i^{-1}, 0) \right] dx
\]

(45)

\textbf{Proof:} The proof of this Lemma, which consists in solving (23) along its characteristics, is given in Appendix B. 

Besides, Lemma 4 is a technical result useful to prove convergence of the Lyapunov function to zero.

\textbf{Lemma 4:} Consider a vector function \( V : t \in \mathbb{R} \mapsto (V_i(t), \ldots, V_n(t))^T \in \mathbb{R}^n \) satisfying

\[
\dot{V}(t) \leq A(t)V(t)
\]

where the inequality is meant component-by-component, \( A = (a_{ij}) \) is lower triangular and its diagonal terms satisfy

\[
\forall t_0 > 0 \int_{t_0}^{t + T} a_{ii}(s) ds < -\epsilon
\]

for some \( \epsilon, T > 0 \). Then, for all \( i = 1, \ldots, n \) one has

\[
\lim_{t \to \infty} V_i(t) = 0.
\]

The proof of this Lemma, which uses the triangular structure of \( A \) to recursively apply Gronwall’s Lemma, is given in Appendix B. Finally, the following technical Lemma is used in the Lyapunov proof.

\textbf{Lemma 5:} For all \( i = 1, \ldots, n \), consider the following signal

\[
f_i(t) = \frac{\lambda_i p_i}{s_i} |v(t, 0)| - k_i |v(t - \lambda_i^{-1}, 0)|,
\]

where \( p_i, s_i \) satisfy (31), (32). Then, \( f_i \) is uniformly bounded and there exists \( \epsilon_i = s_i k_i \lambda_i - \lambda_i p_i \delta > 0 \) such that

\[
\forall t > 0, \int_j^{t + T} f_i(\tau)d\tau \leq \epsilon_i
\]

where \( \delta \) and \( T > 0 \) are defined by the persistence of excitation condition (29).

The proof of this Lemma is given in Appendix B. We are now ready to give the proof of Theorem 1.

\textbf{Proof of Theorem 1:} Differentiating (30) for a given \( i = 1, \ldots, n \) and integrating by parts yields

\[
\dot{V}_i(t) = I_1 + I_2 + I_3 + I_4 + I_5
\]

where

\[
I_1 = p_i \lambda_i |\tilde{\alpha}_i(0)| - \lambda_i p_i e^{-\delta_i |\tilde{\alpha}_i(1)|} - \gamma_i |\tilde{\beta}(\tau_i(0), 0)| - w_i |\tilde{\beta}(0)|
\]

\[
I_2 = -\delta_i \lambda_i p_i \int_0^1 e^{-\delta_i x} |\tilde{\alpha}_i(x)| dx - \delta_i w_i \mu \int_0^1 e^{\gamma_i x} |\tilde{\beta}(x)| dx
\]

\[
I_3 = \lambda_i p_i \int_0^1 e^{-\delta_i x} \text{sign}(\tilde{\alpha}_i(x)) \left( \sigma_{ii}^+ |\tilde{\alpha}_i(x)| + \sigma_{ii}^- |\tilde{\beta}(x)| \right) - \int_x^1 d_{ii}^+(x, \xi) \tilde{\beta}(\xi) d\xi dx
\]

(48)
\(+ w_i \int_0^1 e^{\delta_i x} \text{sign} (\tilde{\beta}(x)) \left( \sigma - \tilde{\beta}(x) \right) \delta i \sum_{i=1}^{i-1} c_{ik} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \hat{V}_k(t - \lambda_i^{-1}) = \dot{\Omega}_i(t) \)

\(I_4 = \sum_{k=1}^{i-1} c_{ik} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \hat{V}_k(t - \lambda_i^{-1}) \)

\(I_5 = -s_i k_i \text{sign} (\delta q_i - \delta q_i) \left( -\dot{q}_i \mid v(t_i(0),0) \right) + \text{sign} (v(t_i(0),0)) e^{-\delta_i x} \hat{u}_i(1) \)

such that

\(\dot{V}_i(t) \leq -\eta_i W_i(t) + f_i(t) s_i |\delta q_i - \delta q_i| + \sum_{k=1}^{i-1} c_{ik} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \hat{V}_k(t - \lambda_i^{-1}) \)

\(\dot{V}_i(t) = -\eta_i W_i(t) + f_i(t) s_i |\delta q_i - \delta q_i| + \sum_{k=1}^{i-1} c_{ik} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \hat{V}_k(t - \lambda_i^{-1}) \)

Moreover, from the definition of \(V_i\), we have

\(s_i |\delta q_i - \delta q_i| = V_i(t) - W_i(t) - \Omega_i(t) \)

\(\dot{V}_i(t) \leq -\left( \eta_i - f_i(t) \right) W_i(t) + \left( f_i(t) V_i(t) \right) \)

\(\dot{V}_i(t) \leq -\eta_i W_i(t) + f_i(t) s_i |\delta q_i - \delta q_i| + \sum_{k=1}^{i-1} c_{ik} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \hat{V}_k(t - \lambda_i^{-1}) \)

\(\dot{V}_i(t) \leq f_i(t) V_i(t) + \sum_{k=1}^{i-1} a_k(t) V_k(t) + \sum_{k=1}^{i-1} b_k(t) V_k(t - \lambda_i^{-1}) \)

\(V(t) = \left( \begin{array} {cc} \dot{V}_1(t) \\ V_1(t-\lambda_2^{-1}) \\ V_2(t) \\ V_1(t-\lambda_3^{-1}) \\ V_2(t-\lambda_3^{-1}) \\ \vdots \\ V_n(t) \end{array} \right) \)

From the hypothesis of persistent excitation (29) there exists \(T\) such that \(f_i(t)\) in (49) satisfies (47), for some \(\varepsilon_i > 0\). Hence applying Lemma 4 to
yields the result.

IV. ADAPTIVE OBSERVER DESIGN FOR PROBLEM 2

In this section we consider the general case in which both the unknowns \( q \) and \( \theta \) are present in (3). Let \( \bar{q}, \bar{\theta} \in \mathbb{R} \) be some initial guesses on parameters \( q \) and \( \theta \), let \( \delta \bar{q} \) and \( \delta \bar{\theta} \) be the estimates of the uncertainties \( \delta q = q - \bar{q} \) and \( \delta \bar{\theta} = \theta - \bar{\theta} \). Let define the scalar operator \( T_a : \mathbb{L}^\infty([0, \infty), \mathbb{R}) \to \mathbb{L}^\infty([0, \infty), \mathbb{R}) \) as

\[
T_a f(t) = f(t) - f(t - a)
\]

Let \( \Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n) \in (0, \infty)^n \) and let \( \Phi = \{ \phi^a \} \Delta \in (0, \infty)^n \) be a family of functions \( \phi^a : [0, \infty) \to \mathbb{R} \) defined as

\[
\phi^a(t) = \delta \bar{q}(t; \Delta) + \delta \bar{\theta}(t; \Delta)
\]

where for estimates \( \delta \bar{q}(t; \Delta) \) and \( \delta \bar{\theta}(t; \Delta) \) we propose the following parametrized update laws

\[
\frac{d}{dt} \delta \bar{q}(t; \Delta) = k_i \text{sign}(T_{\Delta_i}(v(t - \lambda_i^{-1}, 0)))
\]

\[
- k_i \delta \bar{q}(t; \Delta) \left[ T_{\Delta_i}(v(t - \lambda_i^{-1}, 0)) \right]
\]

\[
\frac{d}{dt} \delta \bar{\theta}(t; \Delta) = l_i \left( \hat{u}(t, 1)e^{-\frac{T}{\lambda_i}} + \phi^a(t - \lambda_i^{-1}) \right)
\]

subject to the following initial conditions

\[
\delta \bar{q}(0; \Delta) = 0, \quad \delta \bar{\theta}(0; \Delta) = 0
\]

being \( k_i, l_i > 0 \) some design parameters for \( i = 1, \ldots, n \). Let \( \Delta \in (0, \infty)^n \) and let \( \phi^a \in \Phi \), then we propose the following observer for the system composed by (1)–(3) and the parameters estimation errors \( (\delta \bar{q} - \delta \hat{q}, \delta \bar{\theta} - \delta \hat{\theta}) \)

\[
\hat{u}_t + \Lambda \hat{u}_x = \Sigma^{++} \hat{u} + \Sigma^{+-} \hat{v} + P^+(x)\hat{u}(t, 1)
\]

\[
\hat{v}_t - \mu \hat{v} = \Sigma^{-+} \hat{u} + \sigma^{-} \hat{v} + P^-(x)\hat{u}(t, 1)
\]

with boundary conditions

\[
\hat{u}(t, 0) = \bar{q} \hat{v}(t, 0) + \bar{\theta} + \phi^a(t)
\]

\[
\hat{v}(t, 1) = ru(t, 1) + U(t)
\]

Remark 5: Equations (53)–(56) define a family of observers parametrized by \( \Delta \in (0, \infty)^n \) through \( \phi^a \), therefore each different choice of \( \Delta \) defines a different observer and \( \Delta \) is a degree of freedom of this design.

Remark 6: Note that even in this case we do not estimate directly the parameters but the uncertainties we have on them, for the same motivations expressed in Remark 3.

The following Theorem represents the second main result of this paper and it states the global asymptotic convergence of \((\hat{u}, \hat{v}, \delta \hat{q}, \delta \hat{\theta})\) when observer (53)–(56) is used under certain persistent excitation conditions of signal \( v(t, 0) \).

Theorem 2: Consider the observer (53)–(56). If \( v(t, 0) \) is a bounded signal and there exists \( \Delta \in (0, \infty)^n \) such that

\[
\forall i = 1, \ldots, n \text{ signal } T_{\Delta_i} v(t, 0) \text{ satisfies the following persistent excitation condition}
\]

\[
\exists b, \tilde{b}, T > 0 \quad \text{s.t.} \quad \forall t \in [0, \infty)
\]

\[
\tilde{b} < \int_0^{t+T} \left| T_{\Delta_i} v(z, 0) \right| dz < b
\]

then the estimation errors (57)–(60) and the uncertainties estimate errors \( \delta \hat{q} - \delta \bar{q} \) and \( \delta \hat{\theta} - \delta \bar{\theta} \) converge to zero asymptotically in \( L_1 \) norm.

The proof of Theorem 2, even if technically more involved, is in essence similar to those of Theorem 1 and it is thus presented in Appendix C. The intuition behind the choice of the update laws (51)–(52) for the uncertainties estimates \( \delta \bar{q}, \delta \bar{\theta} \) resides in the following observation. Since the \( \tilde{\theta} \) state of the target system (61)–(64) is independent on the uncertainties, then at the steady state it will converge to zero, regardless the estimation error. In such a situation, from equation (63) and from Lemma 3, we obtain that at the steady state the value of the measurement

\[
\tilde{\theta}(t, 1) = 0
\]
\[\dot{u}_1(t, 1) = \exp\left(\frac{\sigma_1 + \sigma_2}{\lambda_1}\right) \dot{a}_1(t - \lambda_1^{-1}, 0)\]
\[= \exp\left(\frac{\sigma_1 + \sigma_2}{\lambda_1}\right) \left(\delta q T_{\Delta, k} v(t - \lambda_1^{-1}, 0) + \delta\theta_1\right) - \delta\theta_2(t - \lambda_1^{-1})\]

From this expression one can note that since \(\theta_1\) is constant so is \(\delta\theta_1\), and any two successive measurements of \(\hat{u}_1(t, 1)\) will contain the same additional term \(\exp(\sigma_1^+/\lambda_1) \delta\theta_1\) due to the uncertainty on \(\theta_1\). As an immediate consequence we have that for any \(\Delta_1 > 0\) the quantity \(T_{\Delta, \hat{u}_1(t, 1)} = \hat{u}_1(t, 1) - \hat{u}_1(t - \Delta_1, 1)\) will be independent on \(\delta\theta_1\), since the two identical terms cancel out. We can thus think to use \(T_{\Delta, \hat{u}(t, 1)}\) as a input to an estimator \(\delta q_1\). At the steady state indeed \(T_{\Delta, \hat{u}_1(t, 1)}\) will have the following expression
\[T_{\Delta, \hat{u}_1(t, 1)} = \exp\left(\frac{\sigma_1 + \sigma_2}{\lambda_1}\right) \left(\delta q T_{\Delta, \hat{u}_1} v(t - \lambda_1^{-1}, 0) - T_{\Delta, \delta\theta_2}(t - \lambda_1^{-1})\right)\]
and we can thus build an estimator for \(\delta q_1\) by following the same reasoning as in Problem 1, assuming that \(T_{\Delta, v(t, 0)}\) is persistently exciting in the sense of (65). In order to build an estimator for \(\delta\theta_1\) instead, we can apply the same reasoning by directly exploiting the measurement \(\hat{u}_1(t, 1)\). In the steady state expression of \(\hat{u}_1(t, 1)\) we substitute \(\delta q_1\) with its estimate \(\delta q_1(t)\), given by the first estimator. In this way we obtain a cascade structure of the two estimators in which those of \(\delta\theta_1\) is ISS (as the proof of Theorem 2 confirms) relatively to \(\delta\theta_1\) and with respect to the estimation error \(\delta q_1 - \delta q_1\). Therefore as far as the estimate of \(\delta q_1\) converges so does the estimate of \(\delta\theta_1\).

Finally when both the estimates have converged also the state \(\hat{a}_1(t, x)\) will converge and the same reasoning is used to adapt \(\delta\theta_2, \delta\theta_2\) and the other parameters by induction.

V. IMPLEMENTATION REMARKS AND SIMULATIONS

In this section we briefly discuss the degree of freedom appearing in the presented design and we give some examples of the application of the adaptive observers.

A. Degrees of Freedom and Implementation Remarks

This design contains several degrees of freedom.

1) Boundary Conditions in the Kernel Equations:

As mentioned in [8], the boundary conditions (A.73) can arbitrarily be fixed. Its values appear directly in the definition of the observer distributed gains \(P^+\) and \(P^-\) given in (21)–(22).

The effect of these parameters on the transitory is still an open problem and investigating them is out of the scope of this paper.

A possible choice is to chose them so as to ensure continuity of kernels \(M_j\), for \(1 \leq j < i \leq n\).

2) Initial Guess \(q\) of \(\hat{q}\):

The choice of the initial guess \(\hat{q}\) on \(q\) directly affects the boundary conditions (A.72) of the observer gains equations and moreover from the error system boundary condition (13) we have that in case of perfect compensation, i.e. when \(\delta q = \delta q\), the error system behaves as the real system. Therefore since from Theorem 1 we obtain asymptotic stability for any choice of \(q\) it is possible to chose it arbitrarily. A complete characterization on how \(q\) affects the transitory performance is not known, however it can be chosen also in order to simplify the kernel equations, for example setting \(\hat{q} = 0\).

3) Choice of \(\Delta\) When Estimating \(\theta\):

Parameter \(\Delta\) in (50) can be chosen to maximize the persistent excitation properties of signals \(T_{\Delta, v(t, 0)}\) in order to improve the convergence of the observer.

4) Scheduling of the Update Laws:

In order to improve the observer performance during the transitory it is possible to properly schedule the activation of the update laws for the components of \(\delta q, \delta\theta\). In particular we propose the scheduling represented by the functions \(s_i : [0, \infty) \rightarrow \mathbb{R}\) defined as
\[s_i(t) = h(t - t_{s_i}), \quad i = 1, \ldots, n\]
where \(h(t)\) indicates the Heaviside step function, and for \(i = 1, \ldots, n\) the quantities \(t_{s_i}\) are defined by
\[t_{s_i} = \mu^{-1} + \lambda_i^{-1} + 2 \sum_{k=1}^{i-1} \lambda_k^{-1}\]
and we consider the following modified update law
\[\frac{d}{dt} \delta q_i(t) = k_i s_i(t \text{sign}(v(t - \lambda_i^{-1}, 0))) \left(e^{-\gamma_i} - \dot{u}_i(t, 1)\right) + \delta q_i(t - \lambda_i^{-1})(v(t - \lambda_i^{-1}, 0)) - k_i s_i(t) \delta q_i(t) \left|v(t - \lambda_i^{-1}, 0)\right|\]
\[\delta q_i(0) = 0\]

Note that the choice of functions \(s_i(t)\) and of the quantities \(t_{s_i}, i = 1, \ldots, n\) only determine the time instant in which the adaptive part comes into play, but it does not affect the stability properties of the error system. This particular choice for \(t_{s_i}\) is motivated by the fact that only after those time instants the update law (9) starts to make sense due to the cascade structure of the \(\alpha\)-system.

B. Simulation Examples

We consider a \((2 + 1)\) system defined by the following parameters
\[\lambda_1 = 1, \quad \lambda_2 = 2, \quad \mu = 1.5\]
\[\Sigma^+ = \frac{1}{10} \begin{pmatrix} 2 & -4 \\ 1 & -5 \end{pmatrix}, \quad \Sigma^- = \begin{pmatrix} 0.2 \\ -0.2 \end{pmatrix}, \quad \sigma^- = 0.1\]
\[= \begin{pmatrix} 0.2 \\ 0.6 \end{pmatrix}, \quad q = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \quad \theta = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \]
\[r = \begin{pmatrix} 0.2 \\ 2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 0.5 \\ -1 \end{pmatrix}, \quad \hat{\theta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]
Fig. 1. Estimation of parameters $\delta_q$ and $\delta_\theta$.

Fig. 2. State estimation $L_1$ norm with the adaptive scheme.

where the control law

$$U(t) = -ru(t, 1) + 20 \cos \left( \frac{2\pi}{5} t \right)$$

and the choice

$$\Delta = \frac{\lambda_1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

are used to achieve the required persistent excitation conditions.

Fig. 1 shows the time evolution of the estimates of the unknown parameters uncertainties $\delta q$ and $\delta \theta$, while Figs. 2 and 3 show respectively the $L_1$ norm of the state estimation in the cases in which the adaptive observer (53)–(56) and the observer without adaptive part are used.

C. Robustness to Additional Uncertainties

The proposed update law for uncertain parameters relies on delayed measurements. In practice, the value of the delay, which depends on the characteristic velocities of the system will also

Fig. 3. State estimation $L_1$ norm without the adaptive part.

Fig. 4. Parameter estimation with a 10% uncertainty on the characteristic velocities $\lambda_1, \lambda_2, \mu$.

Fig. 5. State estimation with a 10% uncertainty on the characteristic velocities $\lambda_1, \lambda_2, \mu$. 
be uncertain. As pictured on Fig. 4, introducing a 10% error in all of the characteristic velocities may induce potentially large oscillating errors in the parameter estimation. As illustrated on Fig. 5, the state estimator still performs in a relatively satisfactory manner in this case (to be compared with Fig. 3).

VI. CONCLUSIONS AND OUTLOOK

An adaptive observer has been proposed, able to simultaneously estimate states and uncertain boundary parameters from boundary measurements. Future work will focus on two important aspects in view of using this estimator in industrial applications. An important point of focus is to consider more realistic measurements. Indeed, in practical applications, the measurement of the Riemann variables is never available. Rather, sensors measure physical quantities such as pressure or temperature in the drilling example mentioned in the introduction. Near an equilibrium profile, these quantities can be expressed as linear combination of the Riemann variables, but these expressions precisely involve the uncertain parameters estimated in this paper. It is unclear how the presented design will behave in the presence of such additional uncertainties. Further, physical models are often nonlinear, as in the motivating example of multiphase flow in a drilling system mentioned in the introduction. Uncertain boundary parameters will enter the system nonlinearly, and affect the shape of the steady-state profile around which one linearizes. This, in turn, induces uncertainty not only in boundary parameters, but in all the system parameters. Investigating adaptation law for nonlinear boundary conditions is therefore a focus point for future work.

APPENDIX A

OBSERVER KERNELS EQUATIONS

Following the classical backstepping approach, system (11)–(14) is mapped to (17)–(20) by transformation (15), (16) if and only if kernels \( M(x, \xi) \in \mathbb{R}^{n \times n} \) and \( N(x, \xi) \in \mathbb{R}^{n} \) satisfy the following PDEs, defined in the upper triangular domain \( T = \{(x, \xi) | 0 \leq x \leq \xi \leq 1\} \) (for further details the reader is referred to [8]).

for \( 1 \leq i \leq n, \ 1 \leq j \leq n \)

\[ \lambda_i \partial_x M_{ij}(x, \xi) + \lambda_j \partial_x M_{ij}(x, \xi) = \sum_{k=1}^{n} \sigma_{ik}^{++} M_{kj}(x, \xi) + \sigma_{ik}^{+-} N_j(x, \xi) - \sigma_{ij}^{++} M_{ij}(x, \xi) \]

(A.68)

for \( 1 \leq i \leq n \)

\[ -\mu \partial_x N_i(x, \xi) + \lambda_i \partial_x N_i(x, \xi) = \sum_{k=1}^{n} \sigma_{ki}^{++} M_{ki}(x, \xi) + \sigma_{ki}^{+-} N_i(x, \xi) - \sigma_{ii}^{++} N_i(x, \xi) \]

(A.69)

along with the following set of boundary conditions

\[
M_{ij}(x, \xi) = \frac{\sigma_{ij}^{++}}{\lambda_j - \lambda_i} \text{ def } m_{ij}, \quad 1 \leq i, j \leq n, i \neq j \quad (A.70)
\]

\[
N_i(x, \xi) = \frac{\sigma_{ij}^{++}}{\mu_j + \lambda_i} \text{ def } n_i, \quad 1 \leq i \leq n \quad (A.71)
\]

\[
M_{ij}(0, \xi) = \tilde{q}_i N_j(0, \xi) \quad 1 \leq i \leq j \leq n \quad (A.72)
\]

while to ensure the well-posedness we add the following boundary conditions

\[
M_{ij}(x, 1) = m_{ij}, \quad 1 \leq j < i \leq n \quad (A.73)
\]

The well-posedness of these equations follows directly by the proof in [8]. Functions \( D^+: T \to \mathbb{R}^{+} \) and \( d^-: T \to \mathbb{R}^{+} \) satisfy instead the following integral equations

\[
D^+(x, \xi) = M(x, \xi) \Sigma^+ + \int_\chi M(x, \eta)D^+(\eta, \xi) d\eta
\]

\[
d^-(x, \xi) = N(x, \xi) \Sigma^- + \int_\chi N(x, \eta)d^- (\eta, \xi) d\eta
\]

and \( H(x) = \{h_{ij}(x)\}_{1 \leq j < i \leq n} \) is a strict lower triangular matrix such that

\[
h_{ij}(x) = -M_{ij}(0, \xi) + \tilde{q}_i N_j(0, \xi), \quad 1 \leq j < i \leq n
\]

(A.74)

\[
h_{ij}(x) = 0, \quad 1 \leq i \leq j \leq n
\]

(A.75)

Finally, the observer gains are obtained as

\[
P^+(x) = M(x, 1) \Lambda^+
\]

\[
P^-(x) = N(x, 1) \Lambda^+
\]

APPENDIX B

PROOFS OF LEMMAS 3 AND 4

Proof of Lemma 3: The proof follows from the method of the characteristics. For a given time \( t \), we define the \( i^{th} \) characteristic of (23) as follows

\[
\chi(s) = s, \quad \tau_i(s) = \tau_i(t, s) = t - \lambda_i^{-1}(1 - s), \text{ for } s \in [0, 1]
\]

These characteristic lines of the \((t, x)\)-plane originate at \((t - \lambda_i^{-1}, 0)\) and terminate at \((t, 1)\). For a fixed \( t \), consider now the function

\[
\gamma_i(s) = \alpha_i(\tau_i(s), \chi(s))
\]

Plugging into (23) yields

\[
\lambda_i \gamma_i'(s) = \sigma_{ii}^{++} \gamma_i(s) + \sigma_{ii}^{+-} \beta(\tau_i(s), \chi(s))
\]

\[
- \int_{\chi(s)}^{1} d\xi (\chi(s), \xi) \beta(\tau_i(s), \xi) d\xi
\]
Solving this ODE between \( s = 0 \) and \( s = 1 \) yields

\[
\gamma_i(1) = e^{\frac{s_i^+}{s_i^+ - \tilde{\gamma}_i(0)}} \gamma_i(0) + \int_0^1 e^{\frac{s_i^+(s_i-\tilde{\gamma}_i(0))}{s_i^+ - \tilde{\gamma}_i(0)}} \left( \sigma_i^+ - \tilde{\beta}(\tau_i(s), \chi(s)) - \int_{\chi(s)} d^+(\chi(s), \xi) \tilde{\beta}(\tau_i(s), \xi) d\xi \right) ds
\]

Noticing that

\[
\gamma_i(1) = \tilde{\alpha}(t, 1) = \tilde{u}(t, 1)
\]

and, using (25)

\[
\gamma_i(0) = \tilde{\alpha}(\tau_i(0), 0) = \tilde{q}_i \tilde{\beta}(\tau_i(0), 0) + \int_0^1 \sum_{k=1}^{i-1} h_k(\xi) \tilde{\alpha}_k(\tau_i(0), 0) d\xi
\]

yields the result.

**Proof of Lemma 4:** The first component of \( V \) thus satisfies

\[
\dot{V}_1(t) \leq a_{11}(t) V_1(t)
\]

By Gronwall’s inequality, one has

\[
V_1(t) \leq \exp \left( \int_0^t a_{11}(s) ds \right) V_1(0) \quad (B.76)
\]

By using (B.76) on intervals of the form \([nT, (n+1)T] \), \( n \in \mathbb{N} \), one gets that \( V_1 \) is bounded and asymptotically converges to zero. The second term satisfies

\[
\dot{V}_2(t) \leq a_{22}(t) V_2(t) + a_{21}(t) V_1(t)
\]

Consider \( U_2 \) s.t.

\[
\dot{U}_2(t) = a_{22}(t) U_2(t) + a_{21}(t) V_1(t), \quad U_2(0) = V_2(0)
\]

Applying Gronwall’s inequality to \( U_2 \) yields

\[
\dot{V}_2(t) \leq a_{22}(t) V_2(t) + a_{21}(t) V_1(t)
\]

Besides, \( U_2 \) satisfies

\[
U_2(t) = \exp \left( \int_0^t a_{22}(s) ds \right) U_2(0) + \int_0^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds
\]

Let us prove that \( U_2 \) is bounded and converges to zero. First, notice that for all \( t > s > 0 \), and all \( i = 1, \ldots, n \) there exists \( K > 0 \) such that

\[
\exp \left( \int_s^t a_{ii}(u) du \right) < K \exp \left( -\epsilon \frac{t - s}{T} \right) \quad (B.77)
\]

where \( c, T \) are defined by (46). Indeed, let \( n \in \mathbb{N}, \Delta t \in [0, T] \) be such that \( t = s + nT - \Delta t \). This yields

\[
\exp \left( \int_s^t a(u) du \right) = \exp \left( \int_s^{s+nT-\Delta t} a(u) du \right)
\]

\[
= \exp \left( \int_s^{s+nT} a(u) du \right) \exp \left( -\int_{s+nT-\Delta t}^{s+nT} a(u) du \right) \leq \exp(-n\epsilon) K
\]

where \( K = \exp(\max_{s \in \mathbb{R}, \Delta t \in [0, T]} \int_{s+nT-\Delta t}^{s+nT} a(u) du) \). Noticing that \( n = \frac{t - s}{T} + \Delta t \geq \frac{t - s}{T} \) yields (B.77). Let now \( \eta > 0 \). We now prove there exists \( c, \tilde{L} > 0 \) such that for \( t \geq c, |U_2(t)| < \eta \). First, since \( V_1 \) converges to zero and \( a_{21}(\cdot) \) is bounded, there exists \( c > 0 \) such that

\[
\forall s \geq c, \quad |a_{21}(s) V_1(s)| < \eta \quad (B.78)
\]

Let us now denote

\[
I(t) = \int_0^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds
\]

Let \( t \geq c \), one has

\[
I(t) = \exp \left( \int_0^t a_{22}(\tau) d\tau \right) \int_0^c \exp \left( \int_0^s a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds + \int_c^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds
\]

Because of (46), for \( t \) sufficiently large, one has

\[
\exp \left( \int_0^t a_{22}(\tau) d\tau \right) < \eta
\]

Besides, there exists \( M > 0 \) such that

\[
\int_0^c \exp \left( \int_0^s a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds < M
\]

since the bounds of the integral are fixed and all the terms are bounded on \([0, c] \). Finally, using (B.78) yields

\[
\left| \int_c^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds \right| \leq \eta \int_c^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) ds
\]
Using (B.77) yields
\[
\left| \int_t^\infty \exp \left( \int_s^t a_{22}(\tau)d\tau \right) a_{21}(s)V_1(s)ds \right|
\leq \eta \int_t^\infty K \exp \left( -\frac{t - s}{T} \right) ds
\leq \eta KT \frac{T}{\epsilon} \left[ \exp \left( -\frac{t - s}{T} \right) \right]_t
\leq \eta KT \frac{T}{\epsilon} \left( 1 - \exp \left( -\frac{t - c}{T} \right) \right)
\leq \frac{\eta KT}{\epsilon}.
\]
Thus, for \( t \) sufficiently large, one has
\[
|I(t)| < \eta \left( M + \frac{KT}{\epsilon} \right)
\]
which shows that \( U_2(t) \) converges to zero. The result follows by induction.

**APPENDIX C**

**Proof of Theorem 2**

Finally we prove Theorem 2.

**Proof of Theorem 2**: Let us fix \( \Delta \in [0, \infty)^n \) such that for each \( i = 1, \ldots, n \), \( T \Delta, v(t - \lambda_i^{-1}, 0) \) satisfies (65). In this proof all the dependencies from \( \Delta \) and \( t \) are omitted for the sake of readability. Consider the following sequence \( \{V_i\}_{i=1}^n \) of functions \( V_i : [0, \infty) \to \mathbb{R} \) defined as
\[
V_i(t) = p_i \int_0^t e^{-\beta \cdot \eta} |\tilde{\alpha}_i(t, x)|dx + w_i \int_0^t e^{\beta \cdot \eta} |\tilde{\beta}(t, x)|dx
+ r_i \int_0^t e^{\gamma \cdot \eta} |\tilde{\beta}(t - \lambda_i^{-1}(1 - x), x)|dx
+ g_i \int_0^t e^{\gamma \cdot \eta} |T \Delta, \tilde{\beta}(t - \lambda_i^{-1}(1 - x), x)|dx
+ s_i |\delta q_i - \tilde{\delta}_i| + z_i |\delta \theta_i - \tilde{\delta \theta}_i|
\sum_{k=1}^{i-1} c_{ik} V_k(t) + \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda_i^{-1})
\sum_{k=1}^{i-1} e_{ik} V_k(t - \lambda_i^{-1} - \Delta_i)
\]
Then, omitting the time argument and integrating by parts yields
\[
\dot{V}_i(t) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \quad \text{(C.79)}
\]
where
\[
I_1 = \lambda_i p_i |\tilde{\alpha}_i(0)| - \lambda_i p_i e^{-\delta \cdot \eta} |\tilde{\alpha}_i(1)| - w_i |\tilde{\beta}(0)|
- r_i |\tilde{\beta}(t - \lambda_i^{-1}, 0)| - g_i |T \Delta, \tilde{\beta}(t - \lambda_i^{-1}, 0)|
\]
\[
I_2 = -\delta \lambda_i p_i \int_0^t e^{-\delta \cdot \eta} |\tilde{\alpha}_i(x)|dx - \delta w_i \int_0^t e^{\beta \cdot \eta} |\tilde{\beta}(x)|dx
- \gamma_i \mu r_i \int_0^t e^{\gamma \cdot \eta} |\tilde{\beta}(t - \lambda_i^{-1}(1 - x), x)|dx
- \xi_i \mu g_i \int_0^t e^{\gamma \cdot \eta} |T \Delta, \tilde{\beta}(t - \lambda_i^{-1}(1 - x), x)|dx
\]
\[
I_3 = p_i \int_0^t e^{-\delta \cdot \eta} \text{sign} (\tilde{\alpha}_i(x)) \left( \sigma_{i+}^+ \tilde{\alpha}_i(x) + \sigma_i^+ \tilde{\beta}(x) \right)
- \int_x^1 d_i^+(x, \eta)(\tilde{\beta}(\eta)d\eta)
dx
+ w_i \int_0^t e^{\beta \cdot \eta} \text{sign} (\tilde{\beta}(x)) \left( \sigma_i^- \tilde{\beta}(x) \right)
- \int_x^1 d_i^- (x, \eta)(\tilde{\beta}(\eta)d\eta)
dx
+ r_i \int_0^t e^{\gamma \cdot \eta} \text{sign} (\tilde{\beta}(t - \lambda_i^{-1}(1 - x), x))
\cdot \left( \sigma_i^- \tilde{\beta}(t - \lambda_i^{-1}(1 - x), x) \right)
\]
\[
- \int_x^1 d^-(x, \xi)(\tilde{\beta}(t - \lambda_i^{-1}(1 - x), \xi)d\xi)
dx
+ g_i \int_0^t e^{\gamma \cdot \eta} \text{sign} \left( T \Delta, \tilde{\beta}(t - \lambda_i^{-1}(1 - x), x) \right)
\cdot \left( \sigma_i^- T \Delta, \tilde{\beta}(t - \lambda_i^{-1}(1 - x), x) \right)
\]
\[
- \int_x^1 d^-(x, \eta)T \Delta, \tilde{\beta}(t - \lambda_i^{-1}(1 - x), \eta)d\eta)
dx
I_4 = -s_i k_i \text{sign} \left( \delta q_i - \tilde{\delta}_i \right)
\cdot \left( \text{sign} \left( T \Delta, v(t - \lambda_i^{-1}, 0) \right) T \Delta, \tilde{u}_i(1)e^{-\delta \cdot \eta}
+ \text{sign} \left( T \Delta, v(t - \lambda_i^{-1}, 0) \right) T \Delta, \phi_i(t - \lambda_i^{-1})
- \delta q_i(t) \left| T \Delta, v(t - \lambda_i^{-1}) \right| \right)
\]
\[
I_5 = -z_i l_i \text{sign} \left( \delta \theta_i - \tilde{\delta \theta}_i \right) \left( \tilde{u}_i(1)e^{-\delta \cdot \eta}
+ \phi_i(t - \lambda_i^{-1}) \right)
- v(t - \lambda_i^{-1}, 0) \delta q_i(t) - \delta \theta_i(t)
\]
\[
I_6 = \sum_{k=1}^{i-1} c_{ik} \dot{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \dot{V}_k(t - \lambda_i^{-1})
+ \sum_{k=1}^{i-1} e_{ik} \dot{V}_k(t - \lambda_i^{-1} - \Delta_i)
\]
Introducing bounds (39)–(42) we obtain

\[ I_1 \leq (\lambda, \eta, \bar{Q} - w, \mu) |\bar{\beta}(0)| + \lambda, p_i |\delta q_i - \delta q_i| |v(t, 0)| + \lambda, p_i |\delta \theta_i - \delta \theta_i| + \lambda, p_i H e^{\delta} \left| \int_0^1 e^{-\delta \eta} \sum_{k=1}^{i-1} \bar{\alpha}_k(\eta) d\eta \right| \]

\[ - r_i \mu |\bar{\beta}(t - \lambda_i^{-1}, 0)| - g_i \mu |T_{\Delta}, \bar{\beta}(t - \lambda_i^{-1}, 0)| \]

\[ I_3 \leq p_i \Sigma \int_0^1 e^{-\delta \eta} |\bar{\alpha}_i(\eta)| d\eta \]

\[ + (p_i + w_i)(\Sigma + D) \int_0^1 e^{\delta \eta} |\bar{\beta}(\eta)| d\eta \]

\[ + r_i (\Sigma + D) \int_0^1 e^{\gamma_i \eta} |\bar{\beta}(t - \lambda_i^{-1}, (1 - x), \eta)| d\eta \]

\[ + g_i (\Sigma + D) \int_0^1 e^{\gamma_i \eta} |T_{\Delta}, \bar{\beta}(t - \lambda_i^{-1}, (1 - x), \eta)| d\eta \]

and from Lemma 3 we have

\[ \tilde{u}_i(t, 1) e^{-\frac{s_i t}{l_i}} \]

\[ = \tilde{\alpha}_i(t - \lambda_i^{-1}, 0) \]

\[ + \lambda_i^{-1} \int_0^1 e^{-\frac{s_i \eta}{l_i}} \left( \sigma_i^{+} \tilde{\beta}(t - \lambda_i^{-1}, (1 - x), \eta) \right) d\eta \]

\[ - \int_x^1 d\eta(\eta, \xi) \tilde{\beta}(t - \lambda_i^{-1}, (1 - x), \eta) d\xi \]

\[ = \tilde{q}_i \tilde{\beta}(t - \lambda_i^{-1}, 0) + \delta q_i |v(t - \lambda_i^{-1}, 0)| + \delta \theta_i \]

\[ - \phi_i(t - \lambda_i^{-1}) + \int_0^1 \sum_{k=1}^{i-1} h_{ik}(\eta) \bar{\alpha}_k(\eta) d\eta \]

\[ + \lambda_i^{-1} \int_0^1 e^{-\frac{s_i \eta}{l_i}} \left( \sigma_i^{+} \tilde{\beta}(t - \lambda_i^{-1}, (1 - x), \eta) \right) d\eta \]

\[ - \int_x^1 d\eta(\eta, \xi) \tilde{\beta}(t - \lambda_i^{-1}, (1 - x), \eta) d\xi \]

Therefore we obtain, by linearity of \( T_{\Delta} \)

\[ I_5 \leq z_i l_i \bar{Q} |\bar{\beta}(t - \lambda_i^{-1}, 0)| \]

\[ + z_i l_i |\delta q_i - \delta q_i| |v(t - \lambda_i^{-1}, 0)| \]

\[ + z_i l_i H \left| \int_0^1 \sum_{k=1}^{i-1} \bar{\alpha}_k(\eta) d\eta \right| \]

\[ + z_i l_i \lambda_i^{-1} e^{\frac{\xi}{l_i}} (\Sigma + D) \]

\[ \int_0^1 e^{\gamma_i \eta} |\bar{\beta}(t - \lambda_i^{-1}, (1 - x), \eta)| d\eta \]

\[ - z_i l_i |\delta \theta_i - \delta \theta_i| \]

Consider the the following definitions

\[ W_i(t) = V_i(t) - s_i |\delta q_i - \delta q_i| - \sum_{k=1}^{i-1} c_{ik} V_k(t) \]

\[ - \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda_i^{-1}) - \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda_i^{-1} - \Delta_i) \]

\[ = p_i \int_0^1 e^{-\frac{s_i \eta}{l_i}} |\bar{\alpha}_i(\eta)| d\eta + w_i \int_0^1 e^{\delta \eta} |\bar{\beta}(\eta)| d\eta \]

\[ + r_i \int_0^1 e^{\gamma_i \eta} |\bar{\beta}(t - \lambda_i^{-1}, (1 - x), \eta)| d\eta \]

\[ + g_i \int_0^1 e^{\gamma_i \eta} |T_{\Delta}, \bar{\beta}(t - \lambda_i^{-1}, (1 - x), \eta)| d\eta \]

\[ + z_i |\delta \theta_i - \delta \theta_i| \]

\[ f_i(t) = \frac{\lambda_i p_i + z_i l_i}{s_i} |v(t, 0)| - k_i |T_{\Delta}, v(t - \lambda_i^{-1}, 0)| \]

and choices

\[ z_i \geq \frac{\lambda_i}{l_i} p_i \]

\[ w_i \geq \frac{\bar{Q}}{\mu} p_i \]

\[ r_i \geq \frac{\bar{Q}}{\mu} z_i l_i \]

\[ g_i \geq \frac{\bar{Q}}{\mu} s_i k_i \]

\[ \delta_i \geq \max \left\{ \frac{\Sigma p_i + w_i}{w_i \mu} (\Sigma + D) \right\} \]

\[ \gamma_i \geq \frac{1}{\lambda_i \mu} \left( \lambda_i + \frac{z_i l_i}{r_i} e^{\frac{\xi}{l_i}} \right) (\Sigma + D) \]

\[ \xi_i \geq \frac{1}{\lambda_i \mu} \left( \lambda_i + \frac{s_i k_i}{g_i} e^{\frac{\xi}{l_i}} \right) (\Sigma + D) \]
then since for $1 \leq k < i \leq n$ we have, for each $y : [0, \infty) \rightarrow \mathbb{R}$ and for each $\delta_k > 0$

$$\int_0^1 \sum_{k=1}^{i-1} h_{ik}(x) \delta_k(y(t), x) dx$$

$$\leq H \int_0^1 \sum_{k=1}^{i-1} e^{\delta_k} e^{-\delta_k x} |\delta_k(y(t), x)| dx$$

$$\leq H \sum_{k=1}^{i-1} e^{\delta_k} \int_0^1 e^{-\delta_k x} |\delta_k(y(t), x)| dx$$

$$\leq H \sum_{k=1}^{i-1} e^{\delta_k} W_k(y(t))$$

we have that there exists $\eta_i > 0$ such that

$$\dot{V}_i(t) \leq -\eta_i W_i(t) + s_i \left| \delta_{qi} - \delta_qi \right| f_i(t)$$

$$+ \sum_{k=1}^{i-1} \left( \lambda_i p_i H e^{\delta_k} W_k(t) + c_i \dot{V}_k(t) \right)$$

$$+ \sum_{k=1}^{i-1} \left( (s_i k_i + z_i l_i) H e^{\delta_k} W_k(t - \lambda_i^{-1}) + d_i \dot{V}_k(t - \lambda_i^{-1}) \right)$$

$$+ \sum_{k=1}^{i-1} \left( s_i k_i H e^{\delta_k} W_k(t - \lambda_i^{-1} - \Delta_i) + e_i \dot{V}_k(t - \lambda_i^{-1} - \Delta_i) \right)$$

(C.80)

By the hypothesis of persistent excitation of $T_{\Delta_i} v(t - \lambda_i^{-1}, 0)$ we have that $\exists T, h, \hat{b} > 0$ such that relation (65) holds, moreover for all $t \in [0, \infty)$, it is

$$\int_t^{t+T} \left[ T_{\Delta_i} v(y - \lambda_i^{-1}, 0) \right] dy \leq \int_t^{t+T} \left[ v(y - \lambda_i^{-1}, 0) \right] dy$$

$$\leq h \int_t^{t+T} \left[ v(y - \lambda_i^{-1}, 0) \right] dy$$

from which it follows that there exists finite $a_1 > 0$ such that

$$\int_t^{t+T} \left[ v(y - \lambda_i^{-1}, 0) \right] dy \leq a_1$$

and therefore, defined $\zeta(t) = t - \lambda_i^{-1}$ we also have

$$\int_t^{t+T} \left[ v(\zeta(y), 0) \right] dy = \int_{\max(\zeta(t), 0)}^{\max(\zeta(t), 0)} \left[ v(y, 0) \right] dy \leq a_1$$

and thus for the arbitrariness of $t$ in (65) we have that also

$$\int_t^{t+T} \left[ v(y, 0) \right] dy \leq a_1, \quad \forall t \in [0, \infty)$$

In particular we obtain

$$\int_t^{t+T} f_i(y) dy \leq \frac{1}{s_i} \left( a_1 (\lambda_i p_i + z_i l_i) - s_i k_i \hat{b} \right)$$

and therefore with the choice

$$s_i > \max \left\{ \frac{a_1 (\lambda_i p_i + z_i l_i)}{b k_i}, 1 \right\}$$

we obtain

$$\forall t \geq 0, \quad \int_t^{t+T} f_i(y) dy \leq \int_t^{t+T} s_i f_i(y) dy \leq -\epsilon_i$$

being

$$\epsilon_i = s_i k_i \hat{b} - a_1 (\lambda_i p_i + z_i l_i) > 0$$

Furthermore we define

$$\rho_i = \sup_{t \in [0, \infty)} \{ s_i f_i(t) \} (\lambda_i p_i + z_i l_i) \sup_{t \in [0, \infty)} \{ v(t, 0) \}$$

and $\rho_i$ exists finite for each $i = 1, \ldots, n$ by hypothesis of boundedness of $v(t, 0)$. Therefore, from the definition of $W_i$ we have

$$s_i \left| \delta_{qi} - \delta_qi \right| = V_i(t) - W_i(t) - \sum_{k=1}^{i-1} c_k V_k(t)$$

$$- \sum_{k=1}^{i-1} d_i V_k(t - \lambda_i^{-1})$$

and from (C.80) we obtain

$$\dot{V}_i(t) \leq -\omega_i W_i(t) + f_i(t) V_i(t)$$

$$+ \sum_{k=1}^{i-1} \left( \lambda_i p_i H e^{\delta_k} W_k(t) - c_i f_i(t) V_k(t) + c_i \dot{V}_k(t) \right)$$

$$+ \sum_{k=1}^{i-1} \left( (s_i k_i + z_i l_i) H e^{\delta_k} W_k(t - \lambda_i^{-1}) - d_i f_i(t) V_k(t - \lambda_i^{-1}) \right)$$

$$+ \sum_{k=1}^{i-1} \left( s_i k_i H e^{\delta_k} W_k(t - \lambda_i^{-1} - \Delta_i) - e_i \dot{V}_k(t - \lambda_i^{-1} - \Delta_i) \right)$$

(C.81)

where $\omega_i = \eta_i - \rho_i$ can be chosen to be positive for the arbitrariness on $\eta_i$. 

If coefficients $e_{ik}$, $d_{ik}$ and $c_{ik}$ satisfy, for $1 \leq k < i \leq n$

\[
\begin{align*}
    e_{ik} &\geq \frac{\lambda_i p_i H e^{\delta_k}}{\omega_k} \\
    d_{ik} &\geq \frac{(s_i k_i + z_i l_i) H e^{\delta_k}}{\omega_k} \\
    c_{ik} &\geq \frac{s_i k_i H e^{\delta_k}}{\omega_k}
\end{align*}
\]

then from the structure of (C.81) we have that: $V_1$ only depends on $V_1(t)$, $V_2$ depends on $V_1(t)$, $V_2(t)$, $V_3(t)$, $V_1(\tau - \lambda_2^{-1})$ and $V_1(t - \lambda_2^{-1} - \Delta_2)$, $V_3$ depends on $V_1(t)$, $V_2(t)$, $V_3(t)$, $V_1(t - \lambda_2^{-1})$, $V_2(t - \lambda_3^{-1} - \Delta_3)$, $V_1(\tau - \lambda_3^{-1} - \Delta_3)$, $V_1(t - \lambda_2^{-1} - \lambda_3^{-1} - \Delta_3)$, $V_2(t - \lambda_3^{-1} - \Delta_3)$ and $V_1(t - \lambda_2^{-1} - \lambda_3^{-1} - \Delta_3)$, in the same way all the $V_i$ depend on $V_1(t)$ and on opportune delayed versions of $V_i(t)$, with $k < i$, by means of linear combinations whose coefficients are linear functions of quantities $f_i(t)$ delayed opportune. Therefore, if we define vector $V(t)$ as

\[
V(t) = \begin{pmatrix}
    V_1(t) \\
    V_1(t - \lambda_2^{-1}) \\
    V_1(t - \lambda_2^{-1} - \Delta_2) \\
    V_2(t) \\
    V_1(t - \lambda_2^{-1} - \Delta_2) \\
    V_1(t - \lambda_3^{-1}) \\
    V_1(t - \lambda_2^{-1} - \lambda_3^{-1} - \Delta_3) \\
    V_1(t - \lambda_2^{-1} - \lambda_3^{-1}) \\
    V_1(t - \lambda_2^{-1} - \lambda_3^{-1} - \Delta_3) \\
    V_2(t - \lambda_3^{-1} - \Delta_3) \\
    V_2(t - \lambda_3^{-1}) \\
    \vdots \\
    V_3(t) \\
    \vdots \\
    \vdots \\
    \vdots \\
    \vdots \\
    V_n(t) \\
\end{pmatrix}
\]

Then from (C.81) we have that $V$ satisfies

\[
\dot{V}(t) \leq A(t)V(t)
\]

where, again, the inequality is meant component-by-component and $A(t)$ is a square lower triangular matrix whose non null entries are linear combinations of functions $f_i(t)$ opportune delayed, while its diagonal is just $(f_1(t), f_1(t - \lambda_2^{-1}), f_1(t - \lambda_2^{-1} - \Delta_2), f_2(t), \ldots, f_n(t))$. Again, we claim the results using Lemma 4.

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Boundary Estimation of Boundary Parameters for Linear Hyperbolic PDEs

Michelangelo Bin and Florent Di Meglio

Abstract—We propose an adaptive observer scheme to estimate boundary parameters in first-order hyperbolic systems of Partial Differential Equations (PDE). The considered systems feature an arbitrary number of states travelling in one direction and one counter-convecting state. Uncertainties in the boundary reflection coefficients and boundary additive errors are estimated relying on a pre-existing observer design and a novel Lyapunov-based adaptation law.

Index Terms—Kalman filtering, partial differential equations (PDEs).

I. INTRODUCTION

SYSTEMS of first-order hyperbolic Partial Differential Equations (PDEs) are predominant in numerous applications involving transport phenomena, e.g. traffic control [15], open-channel flow [3] or oil production and drilling [13], [6]. Many of these applications feature uncertainties in model parameters that prompt the need for adaptive observer and control design. Certain oil well drilling techniques, referred to as UnderBalanced Drilling, consist in producing oil and gas while drilling, resulting in a multiphase flow inside the well. The dynamic state of this flow, which satisfies a system of first-order hyperbolic PDEs, must be monitored to ensure efficiency and safety of the operation. However, only boundary measurements are typically available in such operations, and the physical properties of the reservoir being drilled are generally poorly known. Estimating boundary (reservoir) parameters is of interest, in this case, not only to improve monitoring of the process but also to collect information on the future productivity of fields.

Typical approaches consist in applying classical estimation approaches, e.g. Kalman filtering [17] or particle filtering [21] to simultaneously estimate states and parameters. These methods, however, have no guarantee of convergence and provide little insight into the observability or identifiability of such systems. In turn, estimation and control schemes for hyperbolic systems have, in a substantial part, been enabled by the design of Lyapunov functions [3], [4] and the backstepping method [11], [20]. Lyapunov functions provide means to study the convergence of the proposed algorithms, while backstepping is a powerful design tool that has widely been used in the context of PDEs (see, e.g. [1], [19]). Another advantage of Lyapunov functions is to enable the design of adaptive schemes, as has been done in the case of parabolic PDEs [12], wave equations [10] or delay systems [2].

In this paper, we consider systems of linear hyperbolic PDEs featuring an arbitrary number of states with positive transport speeds and one state with a negative transport speed. The motivation behind studying such systems lies in the above-mentioned drilling application for which certain multiphase flow models1 have this particularity [16].

In [5], an adaptive observer is presented for this class of systems, able to estimate uncertain additive errors at one boundary. The observer design uses a measurement at the boundary where the uncertainty is located, while the adaptive law uses unmatched measurements. However, this method could not straightforwardly be extended to estimate multiplicative errors (i.e., uncertainty in reflection coefficients). Similarly, in [1] and [7], additive time-varying disturbances are estimated using boundary measurements. Again, the method presented there does not extend to multiplicative uncertainty. Multiplicative uncertainties appear in the aforementioned reservoir characterization problem during drilling, where they result from uncertainty in the reservoir pressure and permeability. In turn, in [8], a single-boundary backstepping observer is designed for hyperbolic systems regardless of the number of states travelling in either direction. Based on this design, we propose here to combine an observer relying on the unmatched measurement with an adaptation law for additive and multiplicative boundary parameters using both boundary measurements. Asymptotic convergence of both state and parameter estimation errors to zero is proved using a Lyapunov function.

The paper is organized as follows. In Section II we define the main objective and briefly recall former results. In Section III we present the main adaptive observer result. In Section V we present simulation results and give some implementation remarks. Most of the technical proofs and developments are contained in Appendices B and C.

1 Namely drift-flux models.
II. PROBLEM DESCRIPTION AND FORMER RESULTS

A. Problem Description

We consider a class of first-order linear hyperbolic systems on the spatial domain \(x \in [0, 1]\) featuring \(n \in \mathbb{N}\) transport equations converging left to right coupled with one transport equation converging right to left. More precisely we consider the following \((n + 1)\) equations of the form

\[
\begin{align*}
\dot{u}_i(t, x) + A u_x(t, x) &= \Sigma^{++} u(t, x) + \Sigma^{+-} v(t, x) \\
\dot{v}_i(t, x) - \mu v_x(t, x) &= \Sigma^{-+} u(t, x) + \Sigma^{--} v(t, x)
\end{align*}
\]

subject to the boundary conditions

\[
\begin{align*}
u(t, 0) &= q \nu(t, 0) + \theta, & v(t, 1) &= r \nu(t, 1) + U(t)
\end{align*}
\]

where \(U(t)\) is the control input, \(q, \theta \in \mathbb{R}^n\) are uncertain constant vectors of boundary parameters and the model coefficients have the following structure

\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 \\
& \ddots \\
0 & \lambda_n
\end{pmatrix}, \quad \mu \in \mathbb{R}_+
\]

with

\[
-\mu < 0 < \lambda_1 < \ldots < \lambda_n
\]

and

\[
\Sigma^{++} = (\sigma_{ij}^+) \quad \Sigma^{--} = (\sigma_{ij}^-), \quad \Sigma^{+-} = \{\sigma_{ij}^{+-}\}_{1 \leq i, j \leq n}
\]

\[
\sigma^{+-} = (\sigma_{1}^{+-} \ldots \sigma_{n}^{+-})^T, \quad \sigma^{--} = \mathbb{R}
\]

\[
q = (q_1 \ldots q_n)^T, \quad \theta = (\theta_1 \ldots \theta_n)^T
\]

\[
r = (r_1 \ldots r_n)
\]

The existence and uniqueness of a solution \(w \in C([0, \infty); H^1((0, 1], \mathbb{R}^{n+m}))\) to (1)–(3) is assessed, e.g. in [18, Theorem 3.1].

Moreover, we consider here for simplicity classical \(C^1\) solutions, which imposes certain compatibility conditions on the initial condition (see e.g. [14]).

Remark 1: Constraint (4) imposes that the system is strictly hyperbolic. This is a necessary condition for the design of a backstepping observer, as illustrated, e.g. in [8] and [9] and as will appear in Equation (A.70).

Remark 2: We consider here constant coupling coefficients and transport velocities for the sake of readability, however the method straightforwardly extends to spatially varying coefficients with more involved technical developments.

In what follows, we solve the two following problems.

Problem 1: Assuming that the following measurement vector is available

\[
y(t) = (u(t, 1)^T v(t, 0))^T,
\]

estimate the states of system (1)–(3) and the uncertain parameter \(q\).

Problem 2: Assuming that the following measurement vector is available

\[
y(t) = (u(t, 1)^T v(t, 0))^T,
\]

estimate the states of system (1)–(3) and the uncertain parameters \(q\) and \(\theta\).

Even if Problem 2 is clearly an extension of Problem 1, we treat the two problems separately. This because estimating at the same time \(q\) and \(\theta\) introduces further technical complications, which make the notation heavier and which are unnecessary when only \(q\) is to be estimated. Moreover solving before Problem 1 provides a softer way to introduce the reader to the more involved Problem 2.

B. Former Results

In [5], an adaptive observer is designed to estimate \(\theta\) in (3), assuming \(q\) perfectly known. The design, there, relies on a matched observer, i.e. an observer relying on the measurement \(v(t, 0)\).

Then, an adaptation law for the estimated parameter \(\hat{\theta}\) is built by computing the transition matrix between the error \(\theta - \hat{\theta}\) and the measurement \(u(t, 1)\), and then performing a gradient descent. This design does not, however, straightforwardly extend to estimating \(q\), since the observer relying on the measurement \(v(t, 0)\) is potentially destabilized by the introduction of an error.

In [8], an observer is designed for general heterodirectional linear hyperbolic systems, i.e. featuring arbitrary numbers of states travelling in each direction. In the context of this paper, this enables the design of an unmatched state observer, i.e. relying on the measurement of \(u(t, 1)\).

In what follows, we rely on such a design and a Lyapunov-based adaptive law using both measurements \(u(t, 1)\) and \(v(t, 0)\) to simultaneously estimate the states and uncertain parameters.

III. ADAPTIVE OBSERVER DESIGN FOR PROBLEM 1

In this section we propose the adaptive observer for system (1)–(3) and for the unknown parameters \(q\). In what follow, we omit the argument \((t, x)\) when not strictly necessary.

A. Observer and Adaptation Law

Let \(\bar{q} \in \mathbb{R}^n\) be the any initial guess on \(q\), which is to remain fixed. We define the quantity \(\delta q\) to be the estimate of the uncertainty \(\delta q = q - \bar{q}\). Following [8], we design an observer of the form

\[
\dot{\hat{u}}_i + \Lambda \dot{\hat{u}}_x = \Sigma^{+} \hat{u} + \Sigma^{-} \hat{v} + P^+(x) \hat{u}(t, 1)
\]

subject to the following boundary conditions

\[
\dot{\hat{v}}_i - \mu \hat{v}_x = \Sigma^{+-} \hat{u} + \Sigma^{-+} \hat{v} + P^-(x) \hat{u}(t, 1)
\]

\[
\hat{u}(t, 0) = \bar{q} \nu(t, 0) + \delta q(t) \nu(t, 0)
\]

\[
\hat{v}(t, 1) = \nu(t, 1) + U(t)
\]

with the following update law for \(\delta q_i, i = 1, \ldots, n\)

\[
\frac{d}{dt} \delta q_i(t) = k_i \text{sign} \left( v(t - \lambda_i^{-1}, 0) \right) \exp \left( -\frac{\Sigma^{+-}}{\lambda_i} \right) \hat{u}_i(t, 1) + \delta q_i(t - \lambda_i^{-1}) v(t - \lambda_i^{-1}, 0) - k_i \delta q_i(t) \left| v(t - \lambda_i^{-1}, 0) \right|
\]
with initial condition
\[ \hat{\delta}q_i(0) = 0 \] (10)

where \( \hat{u}(t, 1) = u(t, 1) - \hat{u}(t, 1), k_i \) for \( i = 1, \ldots, n \) are design parameters and the observer gains \( P^+: [0, 1] \to \mathcal{M}_{n \times n}(\mathbb{R}) \) and \( P^-: [0, 1] \to \mathcal{M}_{1 \times n}(\mathbb{R}) \) are \( L^\infty \) functions chosen according to the design presented in [8] in the case of perfectly known parameters.\(^2\) Defining the distributed errors \( \hat{u}(x, t) = u(x, t) - \hat{u}(x, t) \) and \( \hat{v}(x, t) = v(x, t) - \hat{v}(x, t) \), the observer (5)–(8) leads to the following error system
\[ \begin{align*}
\hat{u}_t + A\hat{u}_x &= \Sigma^+ \hat{u} + \Sigma^- \hat{v} - P^+(x)\hat{u}(t, 1) \\
\hat{v}_t - \mu \hat{v}_x &= \Sigma^- \hat{u} + \sigma^- \hat{v} - P^-(x)\hat{u}(t, 1)
\end{align*} \] (11) (12)

with boundary conditions
\[ \begin{align*}
\hat{u}(t, 0) &= \hat{q}(0, 0) + \delta q(t)v(t, 0) - \hat{q}(t)v(t, 0) \\
\hat{v}(t, 1) &= 0
\end{align*} \] (13) (14)

**Remark 3:** One should notice that we do not directly estimate the unknown parameter \( q \), but the uncertainty \( \delta q \). This because having a time-varying estimate \( \hat{q}(t) \) of \( q \) would lead to kernel equations for \( M(x, \xi) \) and \( N(x, \xi) \) with time-varying boundary conditions (note indeed that \( \hat{g} \) enters in the boundary condition (A.72)). Consequently, this would lead to time-varying observer gains \( P^+(x) \) and \( P^-(x) \). Moreover, this design gives us an additional degree of freedom in the choice of \( \hat{q} \), which is briefly discussed in Section V-A.

**Remark 4:** Note that the first term of (9), namely the term
\[ k_i \exp\left(-\sigma_i^+ \right) \text{sign} (v(t - \lambda_i^{-1}, 0)) \hat{u}_i(t, 1), \]

makes the right-hand side of (9) not continuous in \( t \) since, in general, \( \hat{u}_i(t, 1) \) might not be proportional to \( v(t - \lambda_i^{-1}, 0) \). As a consequence, by directly using (9), classical solutions to (5)–(8) might not necessarily exist. Nevertheless, the potential discontinuities are concentrated in the interval \( [0, t_c] \), with \( t_c := \mu_1^{-1} + \lambda_1^{-1} \). In fact, after \( t_2 := \mu_1^{-1} \), the state \( \beta \) of (23)–(26) is identically null, and consequently, from (25) and from (A.74)–(A.75), one has that for all \( t \geq t_2, \alpha_1(t, 0) = (\delta q_1 - \delta q_1(t))v(t, 0) \). Therefore, from Lemma 3, one obtains that \( \hat{u}_1(t, 1) = \exp(\sigma_{11}^{-1} \lambda_1^{-1})(\delta q_1 - \delta q_1(t))v(t - \lambda_1^{-1}, 0) \) holds for all \( t \geq t_3 \) and thus (9), with \( i = 1 \), is continuous for all \( t \geq t_3 \).

Moreover, equation (25) and Lemma 3 yield
\[ \begin{align*}
\hat{u}_2(t, 1) &= \exp\left(-\sigma_2^+ \right) (\delta q_2 - \delta q_2(t))v(t - \lambda_2^{-1}, 0) \\
&+ \int_0^1 h_1(\xi)\hat{\alpha}_1(t - \lambda_1^{-1}, \xi)d\xi,
\end{align*} \] (15)

and thus, starting from \( t_c = t_3 + \lambda_1^{-1} \), one has that \( \hat{\alpha}_1(t - \lambda_2^{-1}, \xi) \) is proportional to \( v(t - \lambda_2^{-1}, 0) \). As a consequence, for all \( t \geq t_c \), (9) is continuous also for \( i = 2 \). The same reasoning can be extended by induction to all \( i = 3, \ldots, n \), obtaining that, for any \( i = 1, \ldots, n \), (9) is continuous for all \( t \geq t_c \). As a consequence, classical solutions of (5)–(8) can be obtained by scheduling the activation of the update law (9) at any time instant bigger than \( t_c \). The scheduling of the update laws makes sense also for structural reasons, and a particular choice is proposed in Section V-A.

### B. Existence of Observer Gains
For completeness purposes, we briefly recall the following result from [8] regarding the non-adaptive observer design.

**Lemma 1:** Assume \( q \) perfectly known, i.e. \( \delta q = \delta q = 0 \).

There exists functions \( P^+(x) \) and \( P^-(x) \) such that the error system (11)–(14) converges in finite time \( T_0 = \mu_1^{-1} + \sum_{i=1}^{n} \lambda_i^{-1} \) to zero.

**Sketch of the Proof:** The gains are constructed by performing the following backstepping transformation
\[ T : (L^2([0, 1], \mathbb{R}))^{n+1} \to (L^2([0, 1], \mathbb{R}))^{n+1} \]
\[ (\hat{u}, \hat{v}) \to (\hat{\alpha}, \hat{\beta}) \]

with
\[ \begin{align*}
\hat{\alpha}(t, x) &= \hat{u}(t, x) - \int_x^1 M(x, \xi)\hat{\alpha}(t, \xi)d\xi \\
\hat{\beta}(t, x) &= \hat{v}(t, x) - \int_x^1 N(x, \xi)\hat{\alpha}(t, \xi)d\xi
\end{align*} \] (16)

The kernels \( M : T \to \mathcal{M}_{n \times n}(\mathbb{R}) \) and \( N : T \to \mathcal{M}_{1 \times n}(\mathbb{R}) \) are defined on the upper triangular domain \( T = \{ (x, \xi) | 0 \leq x \leq \xi \leq 1 \} \). If they satisfy equations (A.68)–(A.73), then the transformation (15)–(16) maps (11)–(14) into the following target system
\[ \begin{align*}
\hat{\alpha}_1(x) + A\hat{\alpha}_x(x) &= \hat{\Sigma}\hat{\alpha}(x) + \Sigma^+ - \hat{\beta}(x) \\
- \int_x^1 D^+(x, \xi)\hat{\beta}(\xi)d\xi \\
\hat{\beta}_x(x) - \mu \hat{\beta}_x(x) &= \sigma^- \hat{\beta}(x) - \int_x^1 d^-(x, \xi)\hat{\beta}(\xi)d\xi \\
\beta(t, 1) &= 0
\end{align*} \] (17) (18) (19) (20)

where \( \hat{\Sigma} \) is defined as
\[ \hat{\Sigma} = \begin{pmatrix}
\sigma_1^{+1} & & \\
& \ddots & \\
& & \sigma_n^{+1}
\end{pmatrix} \]

and the expressions of functions \( D^+: T \to \mathbb{R}^n, d^- : T \to \mathbb{R} \) and \( H : [0, 1] \to \mathcal{M}_{n \times n}(\mathbb{R}) \) are given in Appendix A. A proof of exponential stability of the target and original systems is given in [8]. The observer gains are then given by
\[ \begin{align*}
P^+(x) &= M(x, 1)A \\
P^-(x) &= N(x, 1)A
\end{align*} \] (21) (22)

where, again, \( M \) and \( N \) are given by (A.68)–(A.73).\(^2\)
C. Target System in the Adaptive Case

The following Lemma gives the expression of the target system in the adaptive case.

**Lemma 2:** Using the same kernels M and N as in the non-adaptive case, transformation (15)–(16) maps system (11)–(14) into the following target system:

\[
\begin{align*}
\dot{\alpha}_t + \Lambda \dot{\alpha}_x &= \bar{\Sigma} \dot{\alpha} + \Sigma^{++} \beta - \int_x^1 D^+(x, \xi) \beta(\xi) d\xi \\
\dot{\beta}_t - \mu \dot{\beta}_x &= \sigma^{--} \beta - \int_x^1 d^-(x, \xi) \beta(\xi) d\xi
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
\alpha(t, 0) &= \tilde{\alpha}(t, 0) + \delta q(t) v(t, 0) + \int_0^1 H(\xi) \tilde{\alpha}(\xi) d\xi \\
\beta(t, 1) &= 0
\end{align*}
\]

where \( \bar{\Sigma} \) is defined as the diagonal of \( \Sigma^{++} \) and the expressions of functions \( D^+: \mathbb{T} \to \mathbb{R}^n, d^-: \mathbb{T} \to \mathbb{R} \) and \( H: [0, 1] \to \mathcal{M}_{n \times n}(\mathbb{R}) \) are given in Appendix A.

**Proof:** Equations (23), (24) and (26) are identical to those derived in the case of known \( q \), since their derivation does not depend on \( q \) but only on \( \tilde{q} \). The only equation changed is the boundary condition of \( \alpha \). Substituting (15)–(16) in (13) indeed yields

\[
\tilde{u}(t, 0) - \tilde{q} \tilde{v}(t, 0) = \delta q(t) v(t, 0) + \int_0^1 M(0, \xi) \tilde{\alpha}(\xi) d\xi
\]

(27)

From equations (A.74)–(A.75) and (A.72) we obtain

\[
\int_0^1 (M(0, \xi) - \tilde{q} N(0, \xi)) \tilde{\alpha}(\xi) d\xi = - \int_0^1 H(\xi) \tilde{\alpha}(\xi) d\xi
\]

(28)

Thus substituting (28) into (27) yields (25).

\[ \blacksquare \]

D. Main Result

We can now state the main result of the paper.

**Theorem 1:** Consider the error system (11)–(14) together with the update law (9) for \( \delta q(t) \), then if \( v(t, 0) \) is bounded and it satisfies the following persistent excitation condition

\[
\exists \bar{b}, \bar{t}, T > 0 \quad \text{s.t.} \quad \forall t \in [0, \infty) \]

\[ b < \int_0^{t+T} |v(\tau, 0)| d\tau < \bar{b} \]

then (11)–(14) and the estimation error \( \delta q - \tilde{\delta} q(t) \) asymptotically converge to zero in \( L_1 \) norm. More precisely, defining, for any \( i = 1, \ldots, n \)

\[
V_i(t) = W_i(t) + \Omega_i(t) + s_i |\delta q_i - \tilde{\delta} q_i(t)|
\]

where

\[
W_i(t) = p_i \int_0^t e^{-\delta \tau} |\tilde{\alpha}_i(\tau, x)| dx + w_i \int_0^t e^{\delta \tau} |\tilde{\beta}_i(\tau, x)| dx
\]

\[ + r_i \int_0^t e^{\delta \tau} |\tilde{\beta}_i(\tau - \lambda_i^{-1}(1 - x), x)| dx \]

and

\[
\Omega_i(t) = \sum_{k=1}^{i-1} c_{ik} V_k(t) + \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda_i^{-1}), \quad \forall i > 1
\]

with appropriate design coefficients \( p_i, w_i, r_i, s_i, c_{ik}, d_{ik} \), for all \( i, k = 1, \ldots, n \) satisfying

\[
p_i > 0 \]

\[
s_i > \max \left\{ \frac{\lambda_i p_i b}{b k_i}, 1 \right\}
\]

\[
w_i \geq \frac{\lambda_i Q M p_i}{\mu}
\]

\[
r_i \geq \frac{s_i k_i Q M}{\mu}
\]

\[
\gamma_i \geq \sigma + D \frac{r_i \lambda_i}{r_i \lambda_i + s_i k_i e^{\frac{x}{\lambda_i}}}
\]

\[
\delta_i \geq \max \left\{ \frac{\sigma}{\lambda_i}, \left( \frac{p_i + w_i}{w_i} \right) \left( \frac{\sigma + D}{w_i} \right) \right\}
\]

\[
c_{ik} \geq \frac{H \lambda_i p_i e^{\delta_i}}{\lambda_k}
\]

\[
d_{ik} \geq \frac{H s_i k_i e^{\delta_i}}{\lambda_k}
\]

where the \( k_i > 0, i = 1, \ldots, n \), appearing in (9) are arbitrary and

\[
\sigma = \max \left\{ \max_{1 \leq i \leq n} |\sigma_i^*|, \max_{1 \leq i \leq n} |\sigma_i^-|, \max_{1 \leq i, j \leq n} |\sigma_{ij}^*|, |\sigma_{ij}^-| \right\}
\]

\[
D = \max \left\{ \sup_{1 \leq i \leq n, (x, \xi) \in \mathbb{T}} |d_i^+(x, \xi)|, \sup_{(x, \xi) \in \mathbb{T}} |d^-(x, \xi)| \right\}
\]

\[
\bar{Q} = \max_{1 \leq i \leq n} |\bar{q}_i|
\]

\[
H = \max_{1 \leq i, j \leq n} \left\{ \sup_{x \in [0, 1]} |h_{ij}(x)| \right\}
\]

(40)

(41)

(42)

are bounds on the system coefficients. Then, one has

\[ \forall i = 1, \ldots, n \quad \lim_{t \to +\infty} V_i(t) = 0. \]

The proof of Theorem 1 involves lengthy and tedious computations: for readability purposes, we first give here the intuition behind the choice of the update law (9). Consider the target system (23), (26). The \( \beta \) state converges to zero after \( t = \mu^{-1} \).
where we omit the time argument for readability when obvious, the convergence of the Lyapunov function to zero. (23) along its characteristics, is given in Appendix B.

which yields the following expression for \( \delta q_1 \)

\[
\delta q_1 = \exp \left( -\frac{\sigma_{ii}^+}{\bar{\lambda}_i} \right) \left( \hat{a}(t) + \delta q_1 (t - \lambda_1^{-1}) v(t - \lambda_1^{-1}, 0) \right)
\]

The update law is then designed as the following modified first-order filter

\[
\frac{d}{dt} \hat{\delta}_1(t) = k_i (\delta q_1 - \hat{\delta}_1) \left| v(t - \lambda_1^{-1}, 0) \right|
\]

Plugging (43) in (44) yields (9). Multiplying the first-order filter by \( \left| v(t - \lambda_1^{-1}, 0) \right| \) ensures that the parameter is no longer updated when excitation is insufficient. Finally, when \( \hat{\delta}_q_1 \) has converged to \( \delta q_1 \), the same reasoning is used to adapt \( \hat{\delta}_q_2 \) and the other parameters by induction. We now give a more formal proof.

**E. Proof of Theorem 1**

The proof relies on the following three Lemmas. Lemma 3 establishes a relation among the measurable boundary signal \( \bar{u}(t, 1) \) and the effect of the uncertainty \( \delta q \) on the system. Lemma 3: For all \( i = 1, \ldots, n \), we define the following time delay operator

\[
\tau_i(t, x) = t - \lambda_i^{-1} (1 - x)
\]

where we omit the time argument for readability when obvious, i.e. \( \tau_i(x) = \tau_i(t, x) \). Consider the error system (11)–(14) and its target system (23)–(26), then for all \( i = 1, \ldots, n \), one has

\[
\bar{u}_i(t, 1) = \exp \left( \frac{\sigma_{ii}^+}{\bar{\lambda}_i} \right) \left( \delta q_i v(\tau_i(0), 0) - \delta q_i (\tau_i(0)) v(\tau_i(0), 0) \right)
\]

Proof: The proof of this Lemma, which consists in solving (23) along its characteristics, is given in Appendix B. Besides, Lemma 4 is a technical result useful to prove convergence of the Lyapunov function to zero.

**Lemma 4**: Consider a vector function \( V : t \in \mathbb{R} \mapsto (V_i(t), \ldots, V_n(t))^T \in \mathbb{R}^n \) satisfying

\[
\dot{V}(t) \leq A(t)V(t)
\]

where the inequality is meant component-by-component, \( A = (a_{ij}) \) is lower triangular and its diagonal terms satisfy

\[
\forall t_0 > 0, \int_{t_0}^{t_0 + T} a_{ii}(s)ds < -\epsilon
\]

for some \( \epsilon, T > 0 \). Then, for all \( i = 1, \ldots, n \) one has

\[
\lim_{t \to \infty} V_i(t) = 0.
\]

The proof of this Lemma, which uses the triangular structure of \( A \) to recursively apply Gronwall’s Lemma, is given in Appendix B. Finally, the following technical Lemma is used in the Lyapunov proof.

**Lemma 5**: For all \( i = 1, \ldots, n \), consider the following signal

\[
f_i(t) = \frac{\lambda_i \bar{p}_i}{s_i} |v(t, 0)| - k_i |v(t - \lambda_i^{-1}, 0)|,
\]

where \( p_i, s_i \) satisfy (31), (32). Then, \( f_i \) is uniformly bounded and there exists \( \epsilon_i = s_i k_i \bar{b} - \lambda_i \bar{p}_i \bar{b} > 0 \) such that

\[
\forall t > 0, \int_0^{t+T} f_i(\tau)d\tau \leq \epsilon_i
\]

where \( \bar{b}, \bar{b} \) and \( T > 0 \) are defined by the persistency of excitation condition (29).

The proof of this Lemma is given in Appendix B. We are now ready to give the proof of Theorem 1.

**Proof of Theorem 1**: Differentiating (30) for a given \( i = 1, \ldots, n \), and integrating by parts yields

\[
\dot{V}_i(t) = I_1 + I_2 + I_3 + I_4 + I_5
\]

where

\[
I_1 = \nu_i \lambda_i |\hat{a}_i(0)| - \lambda_i \nu_i e^{-\delta_i}|\hat{a}_i(1)| - r_i \mu |\bar{\beta}(\tau_i(0), 0)| - w_i \mu |\bar{\beta}(0)|
\]

\[
I_2 = -\delta_i \lambda_i \nu_i \int_0^1 e^{-\delta_i x} |\hat{a}_i(x)|dx - \delta_i \nu_i \mu \int_0^1 e^{\gamma_i x} |\bar{\beta}(x)|dx
\]

\[
I_3 = p_i \int_0^1 e^{-\delta_i x} \text{sign}(\hat{a}_i(x)) \left( \sigma_{ii}^+ \hat{a}_i(x) + \sigma_{ii}^- \bar{\beta}(x) \right)
\]

\[
- \int_x^1 d_i^+(x, \xi) \bar{\beta}(\xi)d\xi \right)dx
\]
Introducing the bounds defined in (39)–(42) yields
\[
I_4 = \sum_{k=1}^{i-1} c_{i_k} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{i_k} \hat{V}_k(t - \lambda_i^{-1}) = \Omega_i(t)
\]
and
\[
I_5 = - s_i k_i \text{sign} \left( \delta q_i - \hat{q}_i \right) \left( - \delta q_i \mid v(\tau_i(0), 0) \right)
+ \text{sign} \left( v(\tau_i(0), 0) \right) e^{-\frac{\tau_i}{\eta_i} - u_i(1)}
+ \hat{q}_i(\tau_i(0)) \mid v(\tau_i(0), 0) \right)
\] (48)

Introducing the bounds defined in (39)–(42) yields
\[
I_1 \leq (p_i \lambda_i Q - w_i \mu) \mid \hat{\beta}(t, 0) \right| + p_i \lambda_i \left| \delta q_i - \hat{q}_i \mid v(t, 0) \right|
- r_i \mu \mid \hat{\beta}(t - \lambda_i^{-1}, 0) \right| + p_i \lambda_i \left| \int_0^{t - 1} h_{i_k}(\xi) \hat{\alpha}_k(\xi) \right|
\]
and
\[
I_3 \leq p_i \sigma \int_0^1 e^{-\beta i x} |\hat{\alpha}_i(x) | dx
+ r_i (\sigma + D) \int_0^1 e^{\gamma x} |\hat{\beta}(\tau_i(x), x) | dx
+ (p_i + w_i) (\sigma + D) \int_0^1 e^{\gamma x} |\hat{\beta}(t, x) | dx
\]

Applying Lemma 3 to (48), i.e., substituting \( \hat{u}_i(1) \) with (45) in (48), yields
\[
I_5 \leq - s_i k_i \left| (\delta q_i - \hat{q}_i) v(t - \lambda_i^{-1}, 0) \right|
+ s_i k_i \hat{Q} \left| \hat{\beta}(t - \lambda_i^{-1}, 0) \right|
+ s_i k_i \left| \int_0^{t_1} \sum_{k=1}^{i-1} h_{i_k}(\xi) \hat{\alpha}_k(\xi) \right|
+ s_i k_i \lambda_i^{-1} e^{\frac{\tau_i}{\eta_i}} (\sigma + D) \int_0^1 e^{\gamma x} |\hat{\beta}(\tau_i(x), x) | dx
\]

Then, with the choices (31) for \( p_i \) and (33)–(36) for \( w_i, r_i, \gamma_i \), and \( \delta_i \), there exists an arbitrarily large constant \( \eta_i(s_i, p_i, K_i) > 0 \)

such that
\[
\dot{V}_i(t) \leq - \eta_i W_i(t) + f_i(t)s_i(\delta q_i - \hat{q}_i(t))
+ \sum_{k=1}^{i-1} c_{i_k} \hat{V}_k(t) + \sum_{k=1}^{i-1} d_{i_k} \hat{V}_k(t - \lambda_i^{-1})
+ \lambda_i p_i \left| \int_0^1 \sum_{k=1}^{i-1} h_{i_k}(\xi) \hat{\alpha}_k(t, \xi) \right|
+ s_i k_i \left| \int_0^1 \sum_{k=1}^{i-1} h_{i_k}(\xi) \hat{\alpha}_k(t - \lambda_i^{-1}, \xi) \right|
\]
\[
\leq - \eta_i W_i(t) + f_i(t)s_i(\delta q_i - \hat{q}_i(t))
+ \sum_{k=1}^{i-1} \left( c_{i_k} \hat{V}_k(t) + H \lambda_i p_i e^{\beta_i} W_k(t) \right)
+ \sum_{k=1}^{i-1} \left( d_{i_k} \hat{V}_k(t - \lambda_i^{-1}) + H s_i k_i e^{\beta_i} W_k(t - \lambda_i^{-1}) \right)
\]
Moreover, from the definition of \( V_i \), we have
\[
s_i(\delta q_i - \hat{q}_i(t)) = V_i(t) - W_i(t) - \Omega_i(t)
\]
and \( \dot{V}_i(t) \) satisfies
\[
\dot{V}_i(t) \leq - (\eta_i - f_i(t)) W_i(t) + f_i(t)V_i(t)
+ \sum_{k=1}^{i-1} \left( c_{i_k} \hat{V}_k(t) - c_{i_k} f_i(t) V_k(t) + H \lambda_i p_i e^{\beta_i} W_k(t) \right)
+ \sum_{k=1}^{i-1} \left( d_{i_k} \hat{V}_k(t - \lambda_i^{-1}) - d_{i_k} f_i(t) V_k(t - \lambda_i^{-1}) \right)
+ H s_i k_i e^{\beta_i} W_k(t - \lambda_i^{-1})
\]
Imposing \( \eta_i > \| f_i \|_\infty \) and picking the \( \delta_i, c_{ij} \) and \( \lambda_i \) and such that (36), (37), (38) hold yields, for some \( a_k(t), b_k(t) \)
\[
\dot{V}_i(t) \leq f_i(t) V_i(t) + \sum_{k=1}^{i-1} a_k(t) V_k(t) + \sum_{k=1}^{i-1} b_k(t) V_k(t - \lambda_i^{-1})
\] (49)

From the hypothesis of persistent excitation (29) there exists \( T \) such that \( f_i(t) \) in (49) satisfies (47), for some \( e_i > 0 \). Hence applying Lemma 4 to
\[
V(t) = \begin{pmatrix}
V_1(t) \\
V_1(t - \lambda_2^{-1}) \\
V_2(t) \\
V_1(t - \lambda_3^{-1}) \\
V_2(t - \lambda_3^{-1}) \\
V_3(t) \\
\vdots \\
V_n(t)
\end{pmatrix}
\]
IV. ADAPTIVE OBSERVER DESIGN FOR PROBLEM 2

In this section we consider the general case in which both the unknowns $\theta$ and $q$ are present in (3). Let $\bar{q}, \bar{\theta} \in \mathbb{R}$ be some initial guesses on parameters $q$ and $\theta$, let $\delta q$ and $\delta \theta$ be the estimates of the uncertainties $\delta q = q - \bar{q}$ and $\delta \theta = \theta - \bar{\theta}$. Let define the scalar operator $T_\Delta : L^\infty([0, \infty), \mathbb{R}) \to L^\infty([0, \infty), \mathbb{R})$ as

$$T_\Delta f(t) = f(t) - f(t-a)$$

Let

$$\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n) \in (0, \infty)^n$$

and let $\Phi = \{\phi^\Delta : \Delta \in (0, \infty)^n\}$ be a family of functions $\phi^\Delta : [0, \infty) \to \mathbb{R}$ defined as

$$\phi^\Delta(t) = \delta q(t; \Delta) v(t, 0) + \delta \theta(t; \Delta)$$

where for estimates $\delta q(t; \Delta)$ and $\delta \theta(t; \Delta)$ we propose the following parametrized update laws

$$\frac{d}{dt} \delta q_i(t; \Delta) = k_i \text{sign} \left(T_\Delta, v(t - \lambda_i^{-1}, 0)\right)$$

$$\cdot T_\Delta \left(\bar{u}(t, 1) e^{-\frac{\lambda_i}{\lambda_i} t} + \phi_i^\Delta(t - \lambda_i^{-1})\right) - k_i \delta q_i(t; \Delta) \left|T_\Delta, v(t - \lambda_i^{-1}, 0)\right|$$

$$\frac{d}{dt} \delta \theta_i(t; \Delta) = l_i \left(\bar{u}_i(t, 1) e^{-\frac{\lambda_i}{\lambda_i} t} + \phi_i^\Delta(t - \lambda_i^{-1})\right)$$

$$- l_i \left(\delta q_i(t; \Delta) v(t - \lambda_i^{-1}, 0) + \delta \theta_i(t; \Delta)\right)$$

subject to the following initial conditions

$$\delta q_i(0; \Delta) = 0, \quad \delta \theta_i(0; \Delta) = 0$$

being $l_i, k_i > 0$ some design parameters for $i = 1, \ldots, n$. Let $\Delta \in (0, \infty)^n$ and let $\phi^\Delta \in \Phi$, then we propose the following observer for the system composed by (1)–(3) and the parameters estimation errors ($\delta q = \delta q_i, \delta \theta = \delta \theta_i$)

$$\hat{u}_t + \Lambda \hat{u}_x = \Sigma^{++} \hat{u} + \Sigma^{-} \hat{v} - P'(x) \hat{u}(t, 1)$$

$$\hat{v}_t - \mu \hat{v}_x = \Sigma^{++} \hat{u} + \Sigma^{-} \hat{v} - P'(x) \hat{u}(t, 1)$$

with boundary conditions

$$\hat{u}(t, 0) = \bar{q}\hat{v}(t, 0) + \bar{\theta} + \phi^\Delta(t)$$

$$\hat{v}(t, 1) = ru(t, 1) + U(t)$$

Remark 5: Equations (53)–(56) define a family of observers parametrized by $\Delta \in (0, \infty)^n$ through $\phi^\Delta$, therefore each different choice of $\Delta$ defines a different observer and $\Delta$ is a degree of freedom of this design.

Remark 6: Note that even in this case we do not estimate directly the parameters but the uncertainties we have on them, for the same motivations expressed in Remark 3.

Defined the distributed errors $\bar{u}(t, x) = u(t, x) - \hat{u}(t, x)$ and $\bar{v}(t, x) = v(t, x) - \hat{v}(t, x)$, the error system reads

$$\bar{u}_t + \Lambda \bar{u}_x = \Sigma^{++} \bar{u} + \Sigma^{-} \bar{v} - P'(x) \bar{u}(t, 1)$$

$$\bar{v}_t - \mu \bar{v}_x = \Sigma^{++} \bar{u} + \Sigma^{-} \bar{v} - P'(x) \bar{u}(t, 1)$$

with boundary conditions

$$\bar{u}(t, 0) = \bar{q}\bar{v}(t, 0) + \delta q v(t, 0) + \delta \theta - \phi^\Delta(t)$$

$$\bar{v}(t, 1) = 0$$

and following the same procedure as in Lemma 2 we find that (57)–(60) is isomorphic through (15)–(16) to the target system

$$\tilde{\alpha}_t + \Lambda \tilde{\alpha}_x = \Sigma \tilde{\alpha} + \Sigma^{-} \tilde{\beta} - \int_{x}^{1} D^+(x, \xi) \tilde{\beta}(t, \xi) d\xi$$

subject to boundary conditions

$$\tilde{\alpha}(t, 0) = \bar{q}\tilde{\beta}(t, 0) + \delta q v(t, 0) + \delta \theta - \phi^\Delta(t)$$

$$+ \int_{0}^{1} H(\xi) \tilde{\alpha}(t, \xi) d\xi$$

$$\tilde{\beta}(t, 1) = 0$$

The following Theorem represents the second main result of this paper and it states the global asymptotic convergence of $(\bar{u}, \bar{v}, \delta q, \delta \theta)$ when observer (53)–(56) is used under certain persistent excitation conditions of signal $v(t, 0)$.

Theorem 2: Consider the observer (53)–(56). If $v(t, 0)$ is a bounded signal and there exists $\Delta \in (0, \infty)^n$ such that

$$\forall i = 1, \ldots, n \text{ signal } T_{\Delta}, v(t, 0) \text{ satisfies the following persistent excitation condition}$$

$$\exists h, \overline{b}, T > 0 \text{ s.t. } \forall t \in [0, \infty)$$

$$b < \int_{t}^{t+T} |T_{\Delta}, v(z, 0)| dz < \overline{b}$$

then the estimation errors (57)–(60) and the uncertainties estimate errors $\delta q = \delta q_i$ and $\delta \theta = \delta \theta_i$ converge to zero asymptotically in $L_1$ norm.

The proof of Theorem 2, even if technically more involved, is in essence similar to those of Theorem 1 and it is thus presented in Appendix C. The intuition behind the choice of the update laws (51)–(52) for the uncertainties estimates $\delta q_i$ and $\delta \theta_i$ resides in the following observation. Since the $\tilde{\beta}$ state of the target system (61)–(64) is independent on the uncertainties, then at the steady state it will converge to zero, regardless the estimation error. In such a situation, from equation (63) and from Lemma 3, we obtain that at the steady state the value of the measurement
\[ \dot{u}_1(t, 1) \text{ will have the expression} \]
\[ \dot{u}_1(t, 1) = \exp \left( \frac{\sigma_{11}^+}{\lambda_1} \right) \dot{\alpha}_1(t - \lambda_1^{-1}, 0) \]
\[ = \exp \left( \frac{\sigma_{11}^+}{\lambda_1} \right) \left( \delta q_1 v(t - \lambda_1^{-1}, 0) + \delta \theta_1 \right) \]
\[ - \dot{\phi}_{q1}^\lambda (t - \lambda_1^{-1}) \]

From this expression one can note that since \( \theta_1 \) is constant so is \( \delta \theta_1 \), and any two successive measurements of \( \dot{u}_1(t, 1) \) will contain the same additional term \( \exp(\sigma_{11}^+ / \lambda_1) \delta \theta_1 \) due to the uncertainty on \( \theta_1 \). As an immediate consequence we have that for any \( \Delta_1 > 0 \) the quantity \( T_{\Delta_1} \dot{u}_1(t, 1) = \dot{u}_1(t, 1) - \dot{u}_1(t - \Delta_1, 1) \) will be independent on \( \delta \theta_1 \), since the two identical terms cancel out. We can thus think to use \( T_{\Delta_1} \dot{u}(t, 1) \) as an input to an estimator \( \delta q_1 \). At the steady state indeed \( T_{\Delta_1} \dot{u}_1(t, 1) \) will have the following expression
\[ T_{\Delta_1} \dot{u}_1(t, 1) = \exp \left( \frac{\sigma_{11}^+}{\lambda_1} \right) \left( \delta q_1 T_{\Delta_1} v(t - \lambda_1^{-1}, 0) \right) \]
\[ - T_{\Delta_1} \dot{\phi}_{q1}^\lambda (t - \lambda_1^{-1}) \]
and we can thus build an estimator for \( \delta q_1 \) by following the same reasoning as in Problem 1, assuming that \( T_{\Delta_1} v(t, 0) \) is persistently exciting in the sense of (65). In order to build an estimator for \( \delta \theta_1 \) instead, we can apply the same reasoning by directly exploiting the measurement \( \dot{u}_1(t, 1) \). In the steady state expression of \( \dot{u}_1(t, 1) \) we substitute \( \delta q_1 \) with its estimate \( \delta \hat{q}_1(t) \), given by the first estimator. In this way we obtain a cascade structure of the two estimators in which those of \( \delta \theta_1 \) is ISS (as the proof of Theorem 2 confirms) relatively to \( \delta \theta_1 \) and with respect to the estimation error \( \delta \theta_1 - \hat{\delta} \theta_1 \). Therefore as far as the estimate of \( \delta \theta_1 \) converges so does the estimate of \( \delta \theta_1 \).

Finally when both the estimates have converged also the state \( \dot{\alpha}_1(t, x) \) will converge and the same reasoning is used to adapt \( \delta \hat{q}_2, \delta \hat{\theta}_2 \) and the other parameters by induction.

V. IMPLEMENTATION REMARKS AND SIMULATIONS

In this section we briefly discuss the degree of freedom appearing in the presented design and we give some examples of the application of the adaptive observers.

A. Degrees of Freedom and Implementation Remarks

This design contains several degrees of freedom.

1) Boundary Conditions in the Kernel Equations:
As mentioned in [8], the boundary conditions (A.73) can arbitrarily be fixed. Its values appear directly in the definition of the observer distributed gains \( P^+ \) and \( P^- \) given in (21)–(22). The effect of these parameters on the transitory is still an open problem and investigating them is out of the scope of this paper.

A possible choice is to chose them so as to ensure continuity of kernels \( M_{ij} \), for \( 1 \leq j \leq i \leq n \).

2) Initial Guess \( \hat{q} \) of \( \hat{q} \):

The choice of the initial guess \( \hat{q} \) on \( \hat{q} \) directly affects the boundary conditions (A.72) of the observer gains equations and moreover from the error system boundary condition (13) we have that in case of perfect compensation, i.e. when \( \delta \hat{q} = \delta q \), the error system behaves as the real system. Therefore since from Theorem 1 we obtain asymptotic stability for any choice of \( \hat{q} \) it is possible to chose it arbitrarily. A complete characterization on how \( \hat{q} \) affects the transitory performance is not known, however it can be chosen also in order to simplify the kernel equations, for example setting \( \hat{q} = 0 \).

3) Choice of \( \Delta \) When Estimating \( \theta \): Parameter \( \Delta \) in (50) can be chosen to maximize the persistent excitation properties of signals \( T_{\Delta_1} v(t, 0) \) in order to improve the convergence of the observer.

4) Scheduling of the Update Laws:
In order to improve the observer performance during the transitory it is possible to properly schedule the activation of the update laws for the components of \( \delta \hat{q}_1 \). In particular we propose the scheduling represented by the functions \( s_i : [0, \infty) \to \mathbb{R} \) defined as
\[ s_i(t) = h(t - t_s), \quad i = 1, \ldots, n \]
where \( h(t) \) indicates the Heaviside step function, and for \( i = 1, \ldots, n \) the quantities \( t_s \) are defined by
\[ t_s = \mu^{-1} + \lambda_4^{-1} + 2 \sum_{k=1}^{i-1} \lambda_k^{-1} \]
and we consider the following modified update law
\[ \frac{d}{dt} \delta \hat{q}_1(t) = k_i s_i(t) \text{sign} (v(t - \lambda_i^{-1}, 0)) \left( e^{-\lambda_i^{-1} t} \dot{u}_i(t, 1) \right) \]
\[ + \delta \hat{q}_1(t - \lambda_i^{-1}) v(t - \lambda_i^{-1}, 0) - k_i s_i(t) \delta \hat{q}_1(t) \left| v(t - \lambda_i^{-1}, 0) \right| \]
\[ \delta \hat{q}_1(0) = 0 \]

Note that the choice of functions \( s_i(t) \) and of the quantities \( t_s, i = 1, \ldots, n \), only determine the time instant in which the adaptive part comes into play, but it does not affect the stability properties of the error system. This particular choice for \( t_s \) is motivated by the fact that only after those time instants the update law (9) starts to make sense due to the cascade structure of the \( \bar{\alpha} \)-system.

B. Simulation Examples

We consider a \((2 + 1)\) system defined by the following parameters
\[ \kappa_1 = 1, \quad \kappa_2 = 2, \quad \mu = 1.5 \]
\[ \Sigma^+ = \frac{1}{10} \begin{pmatrix} -2 & -4 \\ 1 & -5 \end{pmatrix}, \quad \Sigma^- = \begin{pmatrix} 0.2 \\ -0.2 \end{pmatrix}, \quad \sigma^- = 0.1 \]
\[ \Sigma^+ = (0.2, 0.6), \quad q = \begin{pmatrix} 2 \\ -6 \end{pmatrix}, \quad \theta = \begin{pmatrix} 5 \\ -3 \end{pmatrix} \]
\[ r = (0.2, 2), \quad \vec{q} = (0.5, -1), \quad \vec{\theta} = (0, 0) \]
where the control law

\[ U(t) = -ru(t, 1) + 20 \cos \left( \frac{2\pi}{5} t \right) \]

and the choice

\[ \Delta = \frac{\lambda_1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

are used to achieve the required persistent excitation conditions.

Fig. 1 shows the time evolution of the estimates of the unknown parameters uncertainties \( \delta q \) and \( \delta \theta \), while Figs. 2 and 3 show respectively the \( L_1 \) norm of the state estimation in the cases in which the adaptive observer (53)–(56) and the observer without adaptive part are used.

C. Robustness to Additional Uncertainties

The proposed update law for uncertain parameters relies on delayed measurements. In practice, the value of the delay, which depends on the characteristic velocities of the system will also

Fig. 4. Parameter estimation with a 10% uncertainty on the characteristic velocities \( \lambda_1, \lambda_2, \mu \).

Fig. 5. State estimation with a 10% uncertainty on the characteristic velocities \( \lambda_1, \lambda_2, \mu \).
VI. CONCLUSIONS AND OUTLOOK

An adaptive observer has been proposed, able to simultaneously estimate states and uncertain boundary parameters from boundary measurements. Future work will focus on two important aspects in view of using this estimator in industrial applications. An important point of focus is to consider more realistic measurements. Indeed, in practical applications, the measurement of the Riemann variables is never available. Rather, sensors measure physical quantities such as pressure or temperature in the drilling example mentioned in the introduction. Near an equilibrium profile, these quantities can be expressed as linear combination of the Riemann variables, but these expressions precisely involve the uncertain parameters estimated in this paper. It is unclear how the presented design will behave in the presence of such additional uncertainties. Further, physical models are often nonlinear, as in the motivating example of multiphase flow in a drilling system mentioned in the introduction. Uncertain boundary parameters will enter the system nonlinearly, and affect the shape of the steady-state profile around which one linearizes. This, in turn, induces uncertainty not only in boundary parameters, but in all the system parameters. Investigating adaptation law for nonlinear boundary conditions is therefore a focus point for future work.

APPENDIX A

OBSERVER KERNELS EQUATIONS

Following the classical backstepping approach, system (11)–(14) is mapped to (17)–(20) by transformation (15), (16) if and only if kernels \( M(x, \xi) \in \mathbb{R}^n \times n \) and \( N(x, \xi) \in \mathbb{R}^n \) satisfy the following PDEs, defined in the upper triangular domain

\[ T = \{(x, \xi) | 0 \leq x, \xi \leq 1\} \]

for further details the reader is referred to [8]).

for \( 1 \leq i \leq n, 1 \leq j \leq n \)

\[
\lambda_i \frac{\partial}{\partial x} M_{ij}(x, \xi) + \lambda_j \frac{\partial}{\partial \xi} M_{ij}(x, \xi) = \sum_{k=1}^{n} \frac{\sigma_{++}^{k-} M_{kj}(x, \xi)}{\lambda_j - \lambda_i} + \sigma_{-}^{i-} N_{ij}(x, \xi) - \sigma_{++}^{j+} M_{ij}(x, \xi)
\]

(A.68)

for \( 1 \leq i \leq n \)

\[
-\mu \frac{\partial}{\partial x} N_{ii}(x, \xi) + \lambda_i \frac{\partial}{\partial \xi} N_{ii}(x, \xi) = \sum_{k=1}^{n} \frac{\sigma_{k+}^{i-} M_{ki}(x, \xi)}{\mu_i - \lambda_i} + \sigma_{-}^{i-} N_{ii}(x, \xi) - \sigma_{++}^{i+} N_{ii}(x, \xi)
\]

(A.69)

along with the following set of boundary conditions

\[
M_{ij}(x, 0) = \frac{\sigma_{ij}^{++}}{\lambda_j - \lambda_i} \text{def} = m_{ij} \quad 1 \leq i, j \leq n, i \neq j \quad (A.70)
\]

\[
N_{i}(x, 0) = \frac{\sigma_{ij}^{++}}{\mu_i + \lambda_i} \text{def} = n_i \quad 1 \leq i \leq n \quad (A.71)
\]

\[
M_{ij}(0, \xi) = \bar{q}_i N_j(0, \xi) \quad 1 \leq i \leq j \leq n \quad (A.72)
\]

while to ensure the well-posedness we add the following boundary conditions

\[
M_{ij}(x, 1) = m_{ij} \quad 1 \leq j < i \leq n \quad (A.73)
\]

The well-posedness of these equations follows directly by the proof in [8]. Functions \( D^+: T \rightarrow \mathbb{R}^n \) and \( d^-: T \rightarrow \mathbb{R}^n \) satisfy instead the following integral equations

\[
D^+(x, \xi) = M(x, \xi) \Sigma^{++} + \int_{\xi} T M(x, \eta) D^+(\eta, \xi) d\eta
\]

\[
d^-(x, \xi) = N(x, \xi) \Sigma^{++} + \int_{\xi} T N(x, \eta) d^-(\eta, \xi) d\eta
\]

and \( H(x) = \{h_{ij}(x)\}_{1 \leq j < i \leq n} \) is a strict lower triangular matrix such that

\[
h_{ij}(x) = -M_{ij}(0, \xi) + \bar{q}_i N_j(0, \xi) , \quad 1 \leq j < i \leq n \quad (A.74)
\]

\[
h_{ij}(x) = 0, \quad 1 \leq i \leq j \leq n \quad (A.75)
\]

Finally, the observer gains are obtained as

\[
P^+(x) = M(x, 1) A^+
\]

\[
P^-(x) = N(x, 1) A^+
\]

APPENDIX B

PROOFS OF LEMMAS 3 AND 4

Proof of Lemma 3: The proof follows from the method of the characteristics. For a given time \( t \), we define the \( i^{th} \) characteristic of (23) as follows

\[
\chi(s) = s, \quad \tau_i(s) = \tau_i(t, s) = t - \lambda_i^{-1}(1 - s), \quad \text{for} \quad s \in [0, 1]
\]

These characteristic lines of the \((t, x)\)-plane originate at \((t - \lambda_i^{-1}(1), 0)\) and terminate at \((t, 1)\). For a fixed \( t \), consider now the function

\[
\gamma_i(s) = \alpha_i(\tau_i(s), \chi(s))
\]

Plugging into (23) yields

\[
\lambda_i \gamma_i'(s) = \sigma_{i+}^{++} + \gamma_i(s) + \sigma_{i+}^{--} \beta(\tau_i(s), \chi(s))
\]

\[
- \int_{\chi(s)}^{t} d\xi \chi(s, \xi) \beta(\tau_i(s), \xi) d\xi
\]
Solving this ODE between \( s = 0 \) and \( s = 1 \) yields
\[
\gamma_i(1) = e^{\frac{s_1^+}{a_i}} \gamma_i(0) + \int_0^1 e^{\frac{s_1^+(t-s)}{a_i}} \left( \frac{\sigma_i^{t_s} - \beta_i(t, s)}{a_i} \right) ds - \int_0^1 d^+(\gamma_i(s, \xi)) \beta_i(\xi, t, s) \xi d\xi.
\]
Noticing that
\[
\gamma_i(1) = \hat{\alpha}(t, 1) + \hat{a}(t, 1)
\]
and, using (25)
\[
\gamma_i(0) = \hat{\alpha}(\tau_i(0), 0) + \hat{q}_i \hat{\beta}_i(\tau_i(0), 0) + \int_0^1 \frac{1}{2} \sum_{k=1}^{n+1} h_k(\xi) \hat{\alpha}_k(\tau_i(0), \xi) d\xi,
\]
yields the result.

**Proof of Lemma 4:** The first component of \( V \) thus satisfies
\[
\dot{V}_1(t) \leq a_{11}(t) V_1(t)
\]
By Gronwall’s inequality, one has
\[
V_1(t) \leq \exp \left( \int_0^t a_{11}(s) ds \right) V_1(0).
\]
By using (B.76) on intervals of the form \([nT, (n+1)T] \), \( n \in \mathbb{N} \), one gets that \( V_1 \) is bounded and asymptotically converges to zero. The second term satisfies
\[
\dot{V}_2(t) \leq a_{22}(t) V_2(t) + a_{21}(t) V_1(t)
\]
Consider \( U_2 \) s.t.
\[
\dot{U}_2(t) = a_{22}(t) U_2(t) + a_{21}(t) V_1(t), \quad U_2(0) = V_2(0)
\]
Applying Gronwall’s inequality to \( U_2 \) yields
\[
V_2 \leq U_2
\]
Besides, \( U_2 \) satisfies
\[
U_2(t) = \exp \left( \int_0^t a_{22}(s) ds \right) U_2(0) + \int_0^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds
\]
Let us prove that \( U_2 \) is bounded and converges to zero. First, notice that for all \( t > s > 0 \), and all \( i = 1, \ldots, n \) there exists \( K > 0 \) such that
\[
\exp \left( \int_s^t a_{ii}(u) du \right) < K \exp \left( -ct \frac{s}{T} \right)
\]
where \( c, T \) are defined by (46). Indeed, let \( n \in \mathbb{N}, \Delta t \in [0, T] \) be such that \( t = s + nT - \Delta t \). This yields
\[
\exp \left( \int_s^t a(u) du \right) = \exp \left( \int_{s+\Delta t}^{s+nT-\Delta t} a(u) du \right) = \exp \left( \int_s^{s+nT} a(u) du \right) \exp \left( -\int_{s+\Delta t}^{s+nT-\Delta t} a(u) du \right) \leq \exp(-n\epsilon) K
\]
where \( K = \exp(\max_{s \in \mathbb{R}, \Delta t \in [0, T]} \int_{s+\Delta t}^{s+nT-\Delta t} a(u) du) \). Noticing that \( n = \frac{t-s}{T} + \Delta t \geq \frac{t-s}{T} \) yields (B.77). Let now \( \eta > 0 \). We now prove there exists \( c, L > 0 \) such that for \( t \geq c, |U_2(t)| < \eta \).
First, since \( V_1 \) converges to zero and \( a_{21}(\cdot) \) is bounded, there exists \( c > 0 \) such that
\[
\forall s \geq c, |a_{21}(s)V_1(s)| < \eta \quad (B.78)
\]
Let us now denote
\[
I(t) = \int_0^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds
\]
Let \( t \geq c \), one has
\[
I(t) = \exp \left( \int_0^t a_{22}(\tau) d\tau \right) \int_0^c \exp \left( \int_0^s a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds + \int_c^t \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds
\]
Because of (46), for \( t \) sufficiently large, one has
\[
\exp \left( \int_0^t a_{22}(\tau) d\tau \right) < \eta
\]
Besides, there exists \( M > 0 \) such that
\[
\int_0^c \exp \left( \int_0^s a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds < M
\]
since the bounds of the integral are fixed and all the terms are bounded on \([0, c]\). Finally, using (B.78) yields
Using (B.77) yields
\[
\left| \int_t^\infty \exp \left( \int_s^t a_{22}(\tau) d\tau \right) a_{21}(s) V_1(s) ds \right| \\
\leq \eta \int_t^\infty K \exp \left( -\varepsilon \frac{t - s}{T} \right) ds \\
\leq \eta K T \varepsilon \left[ \exp \left( -\frac{t - c}{T} \right) \right]^t \\
\leq \eta K T \varepsilon \left( 1 - \exp \left( -\frac{t - c}{T} \right) \right) \\
\leq \eta \frac{KT}{\varepsilon}
\]

Thus, for \( t \) sufficiently large, one has
\[
|I(t)| \leq \eta \left( M + \frac{KT}{\varepsilon} \right)
\]
which shows that \( U_2(t) \) converges to zero. The result follows by induction. \( \blacksquare \)

APPENDIX C

PROOF OF THEOREM 2

Finally we prove Theorem 2.

Proof of Theorem 2: Let us fix \( \Delta \in [0, \infty)^n \) such that for each \( i = 1, \ldots, n \), \( T_\Delta, v(t - \lambda_i^{-1}, 0) \) satisfies (65). In this proof all the dependencies from \( \Delta \) and \( t \) are omitted for the sake of readability. Consider the following sequence \( \{V_i\}_{i=1}^n \) of functions \( V_i : [0, \infty) \rightarrow \mathbb{R} \) defined as
\[
V_i(t) = p_i \int_0^t e^{-\delta_i x} |\dot{\alpha}_i(t, x)| dx + w_i \int_0^t e^{\beta_i x} |\dot{\beta}(t, x)| dx \\
+ r_i \int_0^t e^{\gamma_i x} |\beta(t - \lambda_i^{-1}(1 - x), x)| dx \\
+ g_i \int_0^t e^{\delta_i x} |T_\Delta \dot{\beta}(t - \lambda_i^{-1}(1 - x), x)| dx \\
+ s_i |\delta q_i - \delta \dot{q}_i| + z_i |\delta \theta_i - \delta \dot{\theta}_i| \\
+ \sum_{k=1}^{i-1} c_{ik} V_k(t) + \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda_i^{-1}) \\
+ \sum_{k=1}^{i-1} e_{ik} V_k(t - \lambda_i^{-1} - \Delta_i)
\]

Then, omitting the time argument and integrating by parts yields
\[
\dot{V}_i(t) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \quad \text{(C.79)}
\]
where
\[
I_1 = \lambda_i p_i |\dot{\alpha}_i(0)| - \lambda_i p_i e^{-\delta_i} |\dot{\alpha}_i(1)| - w_i \mu |\dot{\beta}(0)| \\
- r_i \mu |\dot{\beta}(t - \lambda_i^{-1}, 0)| - g_i \mu |T_\Delta \dot{\beta}(t - \lambda_i^{-1}, 0)| \\
- \delta \dot{q}_i(t) |T_\Delta v(t - \lambda_i^{-1}, 0) - \dot{\alpha}_i(0)| \\
- (\dot{u}_i(1) e^{-\gamma_i}) + g_i \mu |T_\Delta \dot{\beta}(t - \lambda_i^{-1}, 0)| \\
- v(t - \lambda_i^{-1}, 0) \delta q_i(t) - \dot{\theta}_i(t)
\]
\[
I_2 = -\gamma_i \mu r_i \int_0^t e^{\gamma_i x} |\beta(t - \lambda_i^{-1}(1 - x), x)| dx \\
- \xi_i \mu g_i \int_0^t e^{\delta_i x} |T_\Delta \dot{\beta}(t - \lambda_i^{-1}(1 - x), x)| dx \\
I_3 = \rho_i \int_0^t e^{-\delta_i x} |\dot{\alpha}_i(x)| \left( \sigma_{i+}^+ \lambda_i x + \sigma_i^- \dot{\beta}(x) \right) \\dot{\beta}(x) \\
- \int_x^1 d^+ (x, \eta) \dot{\beta}(\eta) d\eta \right) dx \\
+ w_i \int_0^t e^{\delta_i x} \text{sign} \left( \beta(x) \right) \left( \sigma_i^+ \lambda_i + \sigma_i^- \dot{\beta}(x) \right) \\dot{\beta}(x) \\
- \int_x^1 d^+ (x, \eta) \dot{\beta}(\eta) d\eta \right) dx \\
+ r_i \int_0^t e^{\gamma_i x} \text{sign} \left( \beta(t - \lambda_i^{-1}(1 - x), x) \right) \cdot \left( \sigma_i^- \dot{\beta}(t - \lambda_i^{-1}(1 - x), x) \right) \\
- \int_x^1 d^+ (x, \xi) \dot{\beta}(t - \lambda_i^{-1}(1 - x), x) \\
+ g_i \int_0^t e^{\delta_i x} \text{sign} \left( T_\Delta \dot{\beta}(t - \lambda_i^{-1}(1 - x), x) \right) \cdot \left( \sigma_i^+ T_\Delta \lambda_i \dot{\beta}(t - \lambda_i^{-1}(1 - x), x) \right) \\
- \int_x^1 d^+ (x, \eta) T_\Delta \dot{\beta}(t - \lambda_i^{-1}(1 - \eta), \eta) d\eta \right) dx \\
\]

I_4 = -s_i \lambda_i \text{sign} \left( \delta q_i - \delta \dot{q}_i \right) \\
\cdot \left( \text{sign} \left( T_\Delta, v(t - \lambda_i^{-1}, 0) \right) T_\Delta \dot{u}_i(1) e^{-\gamma_i} \right) \\
+ \text{sign} \left( T_\Delta, v(t - \lambda_i^{-1}, 0) \right) T_\Delta \dot{\phi}_i(0) \left( t - \lambda_i^{-1} \right) \\
\cdot \left( \delta \dot{q}_i(t) |T_\Delta v(t - \lambda_i^{-1})| \right) \\
- v(t - \lambda_i^{-1}, 0) \delta q_i(t) - \dot{\theta}_i(t)
\]
\[
I_5 = -z_i \lambda_i \text{sign} \left( \delta \dot{\theta}_i - \delta \dot{\theta}_i \right) \left( \dot{u}_i(1) e^{-\gamma_i} + \phi_i(t - \lambda_i^{-1}) \right) \\
- \left( T_\Delta v(t - \lambda_i^{-1}, 0) \right) \delta q_i(t) - \dot{\theta}_i(t)
\]
\[
I_6 = \sum_{k=1}^{i-1} c_{ik} \dot{V}_k(t) + \sum_{k=1}^{i-1} d_{ik} \dot{V}_k(t - \lambda_i^{-1}) \\
+ \sum_{k=1}^{i-1} e_{ik} \dot{V}_k(t - \lambda_i^{-1} - \Delta_i)
\]
Introducing bounds (39)–(42) we obtain

\[ I_1 \leq (\lambda, p_i \bar{Q} - w_i, \mu)|\tilde{\beta}(0)| + \lambda, p_i|\delta q_i - \delta q_i| |v(t, 0)| \]
\[ + \lambda, p_i|\delta \theta_i - \delta \theta_i| + \lambda, p_i H e^\delta \left| \int_0^1 e^{-\delta \eta} \sum_{k=1}^{i-1} \tilde{a}_k(\eta) d\eta \right| \]
\[ - r_i \mu |\tilde{\beta}(t - \lambda^{-1}_i, 0)| - g_i \mu |T_{\Delta} \tilde{\beta}(t - \lambda^{-1}_i, 0)| \]

\[ I_3 \leq p_i \Sigma \int_0^1 e^{-\delta \eta} |\tilde{a}_i(x)| dx \]
\[ + (p_i + w_i)(\Sigma + D) \int_0^1 e^{\delta \eta} |\tilde{\beta}(x)| dx \]
\[ + r_i (\Sigma + D) \int_0^1 e^{\delta \eta} |\tilde{\beta}(t - \lambda^{-1}_i (1 - x), x)| dx \]
\[ + g_i (\Sigma + D) \int_0^1 e^{\delta \eta} |T_{\Delta} \tilde{\beta}(t - \lambda^{-1}_i (1 - x), x)| dx \]

and from Lemma 3 we have

\[ \tilde{u}_i(t, 1) e^{-\frac{s + \bar{Q}}{\lambda}} = \tilde{a}_i(t - \lambda^{-1}_i, 0) \]
\[ + \lambda^{-1}_i \int_0^1 e^{-\frac{s + \bar{Q}}{\lambda}} \left( \sigma_i \tilde{\beta}(t - \lambda^{-1}_i (1 - x), x) \right) \]
\[ + \lambda^{-1}_i \int_0^1 e^{-\frac{s + \bar{Q}}{\lambda}} \left( \sigma_i \tilde{\beta}(t - \lambda^{-1}_i (1 - x), x) \right) \]
\[ - \int_0^1 d_t^\pm (x, \xi) \tilde{\beta}(t - \lambda^{-1}_i (1 - x), \xi) d\xi \]
\[ = \tilde{q}(t - \lambda^{-1}_i, 0) + \delta q_i, v(t - \lambda^{-1}_i, 0) + \delta \theta_i \]
\[ - \phi_i(t - \lambda^{-1}_i) + \int_0^1 \sum_{k=1}^{i-1} h_i(\eta) \tilde{a}_k(\eta) d\eta \]
\[ + \lambda^{-1}_i \int_0^1 e^{-\frac{s + \bar{Q}}{\lambda}} \left( \sigma_i \tilde{\beta}(t - \lambda^{-1}_i (1 - x), x) \right) \]
\[ - \int_0^1 d_t^\pm (x, \xi) \tilde{\beta}(t - \lambda^{-1}_i (1 - x), \xi) d\xi \]

Therefore we obtain, by linearity of \( T_x \)

\[ I_4 \leq k_i \bar{Q} |T_{\Delta} \tilde{\beta}(t - \lambda^{-1}_i, 0)| \]
\[ + k_i H \left| \int_0^1 \sum_{k=1}^{i-1} T_{\Delta} \tilde{a}_k(t - \lambda^{-1}_i, \eta) d\eta \right| \]
\[ + k_i \lambda^{-1}_i e^\frac{\bar{Q}}{\lambda} (\Sigma + D) \cdot \int_0^1 e^{\delta \eta} |T_{\Delta} \tilde{\beta}(t - \lambda^{-1}_i (1 - x), x)| dx \]
\[ - s_i k_i |\delta q_i - \delta q_i| |T_{\Delta} v(t - \lambda^{-1}_i, 0)| \]

Consider the the following definitions

\[ W_i(t) = V_i(t) - s_i |\delta q_i - \delta q_i| - \sum_{k=1}^{i-1} c_{ik} V_k(t) \]
\[ - \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda^{-1}_i) - \sum_{k=1}^{i-1} d_{ik} V_k(t - \lambda^{-1}_i - \Delta_i) \]
\[ = p_i \int_0^1 e^{-\delta \eta} |\tilde{a}_i(t, x)| dx + w_i \int_0^1 e^{\delta \eta} |\tilde{\beta}(t, x)| dx \]
\[ + r_i \int_0^1 e^{\delta \eta} |\tilde{\beta}(t - \lambda^{-1}_i (1 - x), x)| dx \]
\[ + g_i \int_0^1 e^{\delta \eta} |T_{\Delta} \tilde{\beta}(t - \lambda^{-1}_i (1 - x), x)| dx \]
\[ + z_i |\delta \theta_i - \delta \theta_i| + f_i(t) \]
\[ f_i(t) = \frac{\lambda_i p_i + z_i h_i}{s_i} |v(t, 0)| - k_i |T_{\Delta} v(t - \lambda^{-1}_i, 0)| \]

and choices

\[ z_i \geq \frac{\lambda_i}{l_i} p_i \]
\[ w_i \geq \frac{\lambda_i}{\mu} p_i \]
\[ r_i \geq \frac{\bar{Q}}{\mu} p_i \]
\[ g_i \geq \frac{\bar{Q}}{\mu} s_i k_i \]
\[ \delta_i \geq \max \left\{ \frac{\Sigma}{\lambda_i} \frac{p_i + w_i}{w_i, \mu} (\Sigma + D) \right\} \]
\[ \gamma_i \geq \frac{1}{\lambda_i \mu} \left( \frac{s_i k_i}{g_i} e^{\frac{\bar{Q}}{\lambda}} (\Sigma + D) \right) \]
\[ \xi_i \geq \frac{1}{\lambda_i \mu} \left( \frac{s_i k_i}{g_i} e^{\frac{\bar{Q}}{\lambda}} (\Sigma + D) \right) \]
then since for $1 \leq k < i \leq n$ we have, for each $y : [0, \infty) \rightarrow [0, \infty)$ and for each $\delta_k > 0$

$$\int_0^1 \sum_{k=1}^{i-1} h_{ik}(x) \delta_k(y(t), x) dx$$

$$\leq H \int_0^1 \sum_{k=1}^{i-1} e^{\delta_k - \delta_k x} |\delta_k(y(t), x)| dx$$

$$\leq H \int_0^1 e^{\delta_k} \int_0^1 e^{-\delta_k x} |\delta_k(y(t), x)| dx$$

$$\leq H \int_0^1 e^{\delta_k} W_k(y(t))$$

we have that there exists $\eta_i > 0$ such that

$$\dot{V}_i(t) \leq -\eta_i W_i(t) + s_i \left| \delta_{q_i} - \delta_{q_i} \right| f_i(t)$$

$$+ \sum_{k=1}^{i-1} \left( \lambda_i p_i H e^{\delta_k} W_k(t) + c_i \dot{V}_k(t) \right)$$

$$+ \sum_{k=1}^{i-1} \left( s_i k_i + z_i l_i \right) H e^{\delta_k} W_k(t - \lambda_i^{-1})$$

$$+ d_i \dot{V}_k(t - \lambda_i^{-1})$$

$$+ c_i \dot{V}_k(t - \lambda_i^{-1} - \Delta_i)$$

(C.80)

In particular we obtain

$$\int_t^{t+T} f_i(y) dy \leq \frac{1}{s_i} \left( a_i (\lambda_i p_i + z_i l_i) - s_i \bar{b} \right)$$

and therefore with the choice

$$s_i > \max \left\{ \frac{a_i (\lambda_i p_i + z_i l_i)}{\bar{b}}, 1 \right\}$$

we obtain

$$\forall t \geq 0, \quad \int_t^{t+T} f_i(y) dy \leq \int_t^{t+T} s_i f_i(y) dy \leq -\epsilon_i$$

being

$$\epsilon_i = s_i \bar{b} - a_i (\lambda_i p_i + z_i l_i) > 0$$

Furthermore we define

$$\rho_i = \sup_{t \in [0, \infty)} s_i f_i(t) = \lambda_i p_i + z_i l_i \sup_{t \in [0, \infty)} |v(t, 0)|$$

and $\rho_i$ exists finite for each $i = 1, \ldots, n$ by hypothesis of boundedness of $[v(t, 0)]$. Therefore, from the definition of $W_i$ we have

$$s_i \left| \delta_{q_i} - \delta_{q_i} \right| = V_i(t) - W_i(t) - \sum_{k=1}^{i-1} c_i \dot{V}_k(t)$$

$$- \sum_{k=1}^{i-1} d_i \dot{V}_k(t - \lambda_i^{-1})$$

$$- \sum_{k=1}^{i-1} d_i \dot{V}_k(t - \lambda_i^{-1} - \Delta_i)$$

and from (C.80) we obtain

$$\dot{V}_i(t) \leq -\omega_i W_i(t) + f_i(t) V_i(t)$$

$$+ \sum_{k=1}^{i-1} \left( \lambda_i p_i H e^{\delta_k} W_k(t) - c_i \dot{V}_k(t) \right)$$

$$+ \sum_{k=1}^{i-1} \left( s_i k_i + z_i l_i \right) H e^{\delta_k} W_k(t - \lambda_i^{-1})$$

$$- d_i \dot{V}_k(t) V_k(t - \lambda_i^{-1})$$

$$+ d_i \dot{V}_k(t - \lambda_i^{-1} - \Delta_i)$$

$$+ \sum_{k=1}^{i-1} s_i k_i H e^{\delta_k} W_k(t - \lambda_i^{-1} - \Delta_i)$$

$$- e_i \dot{V}_k(t) V_k(t - \lambda_i^{-1} - \Delta_i)$$

$$+ e_i \dot{V}_k(t - \lambda_i^{-1} - \Delta_i)$$

(C.81)

where $\omega_i = \eta_i - \rho_i$ can be chosen to be positive for the arbitrariness on $\eta_i$. 

By the hypothesis of persistent excitation of $T_{\Delta, v(t - \lambda_i^{-1}, 0)}$ we have that $\exists T, \bar{b} > 0$ such that relation (65) holds, moreover

$$\forall t \in [0, \infty), \text{ it is}$$

$$\int_t^{t+T} [T_{\Delta, v(t - \lambda_i^{-1}, 0)}] dy \leq \int_t^{t+T} [v(y - \lambda_i^{-1}, 0)] dy$$

$$+ \int_t^{t+T} [v(y - \lambda_i^{-1} - \Delta_i, 0)] dy$$

$$\leq \bar{b}$$

from which it follows that there exists finite $a_1 > 0$ such that

$$\int_t^{t+T} [v(y - \lambda_i^{-1}, 0)] dy \leq a_1$$

and therefore, defined $\zeta(t) = t - \lambda_i^{-1}$ we also have

$$\int_t^{t+T} [v(\zeta(y), 0)] dy = \int_{\max(\zeta(t), 0)}^{\zeta(t+T)} [v(y, 0)] dy \leq a_1$$

and thus for the arbitrariness of $t$ in (65) we have that also

$$\int_t^{t+T} [v(y, 0)] dy \leq a_1, \quad \forall t \in [0, \infty)$$
If coefficients $c_{ik}$, $d_{ik}$ and $e_{ik}$ satisfy, for $1 \leq k < i \leq n$

\[
\begin{align*}
    e_{ik} &\geq \frac{\lambda_k p_i H e_{ik}}{\omega_k} \\
    d_{ik} &\geq \frac{(s_i k_i + z_i l_i) H e_{ik}}{\omega_k} \\
    c_{ik} &\geq s_i k_i H e_{ik} / \omega_k
\end{align*}
\]

then from the structure of (C.81) we have that: $V_1$ only depends on $V_1(t)$, $V_2$ depends on $V_1(t)$, $V_2(t)$, $V_1(t - \lambda_2^{-1})$ and $V_1(t - \lambda_2^{-1} - \Delta_2)$, $V_3$ depends on $V_1(t)$, $V_2(t)$, $V_3(t)$, $V_1(t - \lambda_3^{-1})$, $V_2(t - \lambda_3^{-1} - \Delta_3)$, $V_1(t - \lambda_3^{-1} - \lambda_3^{-1} - \Delta_2 - \Delta_3)$, $V_2(t - \lambda_3^{-1} - \lambda_3^{-1} - \Delta_2 - \lambda_3^{-1})$ and $V_1(t - \lambda_2^{-1} - \lambda_3^{-1} - \Delta_2 - \Delta_3)$) In the same way all the $V_i$ depend on $V_i(t)$ and on opportune delayed versions of $V_i(t)$, for $k < i$, by means of linear combinations whose coefficients are linear functions of quantities $f_k(t)$ delayed opportune. Therefore, if we define vector $V(t)$ as

\[
V(t) = \begin{pmatrix}
V_1(t) \\
V_2(t) \\
V_3(t) \\
\vdots \\
V_n(t)
\end{pmatrix}
\]

Then from (C.81) we have that $V$ satisfies

\[
\dot{V}(t) \leq A(t)V(t)
\]

where, again, the inequality is meant component-by-component and $A(t)$ is a square lower triangular matrix whose non null entries are linear combinations of functions $f_k(t)$ opportune delayed, while its diagonal is just $(f_1(t), f_1(t - \lambda_2^{-1}), f_1(t - \lambda_2^{-1} - \Delta_2), f_2(t), \ldots, f_n(t))$. Again, we claim the results using Lemma 4.

**REFERENCES**


